A PROOF OF THE ERDOS-TURAN CONJECTURE ON ASYMPTOTIC ADDITIVE BASES

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Abstract. In this article we present a full proof of the Erdos-Turan conjecture on asymptotic additive bases. The conjecture states that if a set B of non-negative integers is an asymptotic base of natural numbers of order two and we denote the sequence which counts the number of ways that a natural number N can be written as sum of two elements of the set B as $R_B(N)$, then we have that $\limsup_{N\to\infty} R_B(N) = \infty$.

1. Introduction

By [2], in 1937 I. M. Vinogradov succeeded in solving the original Goldbach problem for all odd N which are sufficiently large. Before him, G. H. Hardy and J. E. Littlewood made a serious attack by means of a then new circle method. In the article [1] of Erdos and Turan in 1941 the Erdos-Turan conjecture is stated. It states that for every set B of non-negative integers which is an asymptotic base of the natural numbers of order two, the sequence which counts the number of ways that a natural number N is written as a sum of two elements of the set B, is unbounded.

In this paper we prove Erdos-Turan conjecture. Here, we denote as $R_B(N)$ the sequence which counts the number of ways that a natural number N is written as a sum of two elements of the set B. We use only the starting point of the circle method and using analytical tools we present a full proof of this conjecture.

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2. Preliminary Results

Write $R_B(N)=\sum_{\substack{(n,m)\in B^2\\ n+m=N}} 1$. Write also $B(t)=\sum_{n\in B, n\leq t} 1$.

Lemma 2.1. *Let B be an infinite set of natural numbers. Then,*

For every $\rho\in(0,1)$, we have

$$R_B(N)=\frac{1}{2\pi i}\oint_{|z|=\rho}\frac{\log^2 z}{z^{N+1}}\int_1^\infty\int_1^\infty B(t)B(s)z^{t+s}dtds\ dz$$

Proof. Take the complex analytic function $f(z)=\sum_{n\in B} z^n$ for $|z|<1$. It is easy to see that the powerseries has radius of convergence 1. Then, we have

$$f^2(z)=\sum_{n\in B}\sum_{m\in B} z^{n+m}=\sum_{k=1}^{\infty}(\sum_{\substack{(n,m)\in B^2\\ n+m=k}} 1)z^k=\sum_{k=1}^{\infty} R_B(k)z^k$$

By the Cauchy integral formula, we get

$$R_B(N)=\frac{1}{2\pi i}\oint_{|z|=\rho}\frac{f^2(z)}{z^{N+1}}dz$$

This holds for every $\rho\in(0,1)$. Let a complex number z eith $|z|=\rho\in(0,1)$.
Then $z=\rho e^{i\theta}$, where $\theta\in(-\pi,\pi]$. Let $f_k(z)=\sum_{n=1}^{k}\chi_B(n)z^n$. Here, the function χ_B is the characteristic function of the set B.
We use Abel's summation formula which can be found in [3]. We have,

$$f_k(z)=(\sum_{n=1}^{k}\chi_B(n))\rho^k e^{ik\theta}-\int_1^k\sum_{n\leq t}\chi_B(n)(\rho^t e^{i\theta t})'dt=$$

$$=B(k)\rho^k e^{ik\theta}-\int_1^k B(t)(\rho^t e^{i\theta t})'dt=$$

$$=B(k)\rho^k e^{ik\theta}-\int_1^k B(t)(\rho^t\log\rho\ e^{i\theta t}+\rho^t e^{i\theta t}i\theta)dt=$$

$$B(k)\rho^k e^{ik\theta}-\int_1^k B(t)\rho^t e^{i\theta t}\log(\rho e^{i\theta})dt=$$

$$B(k)z^k-\log z\int_1^k B(t)z^t dt$$

Here, we use the branch of logarithm that is real on every positive real number.

We adopt the convention that $B(t) = 0$ for every $t < 1$.

Therefore, using the fact that $\lim_{k\to\infty} B(k)z^k = 0$(since $|z| < 1$ and $0 \le B(k) \le k$), we get

$$f(z) = \lim_{k\to\infty} f_k(z) = -\log z \int_1^\infty B(t)z^t dt$$

Therefore,

$$f^2(z) = \log^2 z \int_1^\infty \int_1^\infty B(t)B(s)z^{t+s} dtds$$

Thus, we have

$$R_B(N) = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{\log^2 z}{z^{N+1}} \int_1^\infty \int_1^\infty B(t)B(s)z^{t+s} dtds\ dz$$

This completes the proof. □

Proposition 2.2. *Let*

$$\Lambda(x) = -\frac{4\pi^2 \cos(x\pi)}{x} - \frac{4\pi(\cos(\pi x) - \sin(\pi x))}{x^2} + \frac{4\sin(\pi x)}{x^3}$$

Then, we have

$$R_B(N) = \frac{1}{2\sqrt{2}\pi} \int_{2-N}^\infty \int_{t+s=\lambda\ t,s\ge 1-\frac{N}{2}} B(t+\frac{N}{2})B(s+\frac{N}{2}) dH^1(t,s) e^{-\pi\lambda}\Lambda(\lambda) d\lambda$$

Here H^1 stands for the one dimensional Hausdorff measure in $\mathbb{R}^2$.

Proof. From the previous Lemma, we have that

$$R_B(N) = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{\log^2 z}{z^{N+1}} \int_1^\infty \int_1^\infty B(t)B(s)z^{t+s} dtds\ dz$$

Therefore, we get

$$R_B(N) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{\log^2(\rho e^{i\theta})(\int_1^\infty B(t)\rho^t e^{it\theta} dt)^2 i\rho e^{i\theta}}{\rho^{N+1} e^{i(N+1)\theta}} d\theta =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(\log\rho + i\theta)^2 (\int_1^\infty B(t)\rho^t e^{it\theta} dt)^2}{\rho^N e^{iN\theta}} d\theta =$$

$$= \frac{1}{2\pi\rho^N} \int_{-\pi}^{\pi} (\log\rho + i\theta)^2 (\int_1^\infty B(t)\rho^t e^{i(t-\frac{N}{2})\theta} dt)^2 d\theta =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log\rho + i\theta)^2 (\int_{1-\frac{N}{2}}^\infty B(t+\frac{N}{2})\rho^t e^{it\theta} dt)^2 d\theta =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\log\rho + i\theta)^2\int_{[1-\frac{N}{2},\infty)^2} B(t+\frac{N}{2})B(s+\frac{N}{2})\rho^{t+s}e^{i(t+s)\theta}d(t,s)d\theta$$

Now, since the function $(\log\rho + i\theta)^2 B(t+\frac{N}{2})B(s+\frac{N}{2})\rho^{t+s}e^{i(t+s)\theta}$ is integrable for $(t,s,\theta)\in[1-\frac{N}{2},\infty)^2\times[-\pi,\pi]$, by Fubini Theorem we have

$$R_B(N) = \frac{1}{2\pi}\int_{[1-\frac{N}{2},\infty)^2} B(t+\frac{N}{2})B(s+\frac{N}{2})\rho^{t+s}\int_{-\pi}^{\pi}(\log\rho + i\theta)^2 e^{i(t+s)\theta}d\theta d(t,s)$$

This holds for every $\rho\in(0,1)$. We set $\rho = e^{-\pi}$ and we have

$$R_B(N) = \frac{1}{2\pi}\int_{[1-\frac{N}{2},\infty)^2} B(t+\frac{N}{2})B(s+\frac{N}{2})e^{-\pi(t+s)}\int_{-\pi}^{\pi}(-\pi + i\theta)^2 e^{i(t+s)\theta}d\theta d(t,s)$$

Now, we have

$$Re(\int_{-\pi}^{\pi}(-\pi+i\theta)^2 e^{i(t+s)\theta}d\theta) = \int_{-\pi}^{\pi}Re((-\pi+i\theta)^2 e^{i(t+s)\theta})d\theta$$

$$= \int_{-\pi}^{\pi}(\pi^2-\theta^2)\cos((t+s)\theta) + 2\pi\theta\sin((t+s)\theta)d\theta$$

Therefore, we get

$$R_B(N) = \frac{1}{2\pi}\int_{[1-\frac{N}{2},\infty)^2} B(t+\frac{N}{2})B(s+\frac{N}{2})e^{-\pi(t+s)}\int_{-\pi}^{\pi}(\pi^2-\theta^2)\cos((t+s)\theta)+2\pi\theta\sin((t+s)\theta)d\theta d(t,s)$$

Also, we have

$$\int_{-\pi}^{\pi}\pi^2\cos((t+s)\theta)d\theta = \frac{\pi^2}{t+s}(\sin(\pi(t+s) - \sin(-\pi(t+s))) = 2\pi^2\frac{\sin(\pi(t+s))}{t+s}$$

By integration by parts we get

$$\int_{-\pi}^{\pi}\theta^2\cos((t+s)\theta)d\theta = 2\pi^2\frac{\sin((t+s)\pi)}{t+s} - \frac{2}{t+s}\int_{-\pi}^{\pi}\theta\sin((t+s)\theta)d\theta =$$

$$= 2\pi^2\frac{\sin((t+s)\pi)}{t+s} + \frac{2}{t+s}(2\pi\frac{\cos(\pi(t+s))}{t+s} - \frac{1}{t+s}\int_{-\pi}^{\pi}\cos((t+s)\theta d\theta) =$$

$$= 2\pi^2\frac{\sin((t+s)\pi)}{t+s} + \frac{4\pi}{(t+s)^2}\cos(\pi(t+s)) - \frac{4}{(t+s)^3}\sin((t+s)\pi)$$

Therefore, we have

$$\int_{-\pi}^{\pi}(\pi^2-\theta^2)\cos((t+s)\theta)d\theta = -\frac{4\pi}{(t+s)^2}\cos(\pi(t+s)) + \frac{4}{(t+s)^3}\sin((t+s)\pi)$$

Also, easilly we can see that

$$\int_{-\pi}^{\pi} \theta \sin((t+s)\theta) d\theta = \frac{2}{(t+s)^2} \sin((t+s)\pi) - 2\pi \frac{\cos((t+s)\pi)}{t+s}$$

Therefore, we see that

$$R_B(N) = \frac{1}{2\pi} \int_{[1-\frac{N}{2},\infty)^2} B(t+\frac{N}{2}) B(s+\frac{N}{2}) e^{-\pi(t+s)} \Lambda(t+s) d(t,s)$$

By integration on level sets(which can be found in [5]), we can see that

$$R_B(N) = \frac{1}{2\sqrt{2}\pi} \int_{2-N}^{\infty} \int_{t+s=\lambda \ t,s\geq 1-\frac{N}{2}} B(t+\frac{N}{2}) B(s+\frac{N}{2}) dH^1(t,s) e^{-\pi\lambda} \Lambda(\lambda) d\lambda$$

This completes the proof. □

Let

$$B^*(\lambda, N) = \int_{t+s=\lambda \ t,s\geq 1-\frac{N}{2}} B(t+\frac{N}{2}) B(s+\frac{N}{2}) dH^1(t,s)$$

We have

$$B^*(\lambda, N) = \sqrt{2} \int_{1}^{\lambda+N-1} B(t) B(\lambda - t + N) dt = B^*(\lambda + N)$$

for the function $B^*(x) = \int_1^{x-1} B(t)B(x-t)dt$. We continue with the following Lemma.

Lemma 2.3. *Assume that* $c_1\sqrt{x} \leq B(x) \leq c_2\sqrt{x}$ *for* $x \geq M_0$*, where* M_0 *is the least element of the set* B*. Then, there exist* $w_1, w_2 > 0$ *such that*

$$w_1 x^2 \leq B^*(x) \leq w_2 x^2$$

for $x > 2M_0$*.*

Proof. We have

$$\int_{M_0}^{x-M_0} \sqrt{t}\sqrt{x-t} dt = \int_{M_0}^{x-M_0} \sqrt{\frac{x^2}{4} - (t-\frac{x}{2})^2} dt =$$

$$c^2 \frac{x}{2} \int_{M_0}^{x-M_0} \sqrt{1-(\frac{2t}{x}-1)^2} dt = c^2 \frac{x^2}{4} \int_{\frac{2M_0}{x}-1}^{1-\frac{2M_0}{x}} \sqrt{1-t^2} dt =$$

$$\frac{c^2x^2}{8}((1-\frac{2M_0}{x})\sqrt{\frac{4M_0}{x} - \frac{4M_0}{x^2}}) + \arcsin(1-\frac{2M_0}{x})) - c^2\frac{x^2}{8}((\frac{2M_0}{x}-1)\sqrt{\frac{4M_0}{x} - \frac{4M_0}{x^2}} + \arcsin(\frac{2M_0}{x}-1)) =$$

$$c^2 \frac{x^2}{4} \arcsin(1-\frac{2M_0}{x}) + O(x^{\frac{3}{2}})$$

By assumption we can see easily that there exist $w_1, w_2 > 0$ such that

$$w_1 x^2 \leq B^*(x) \leq w_2 x^2$$

for $x > 2M_0$. This completes the proof. □

3. Results towards the final proof

We have

$$R_B(N) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B^*(\lambda+N)e^{-\pi\lambda}\Lambda(\lambda)d\lambda$$

By changing variables, we get

$$R_B(N) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B^*(\lambda)e^{-\pi(\lambda-N)}\Lambda(\lambda-N)d\lambda = e^{\pi N}\frac{1}{2\pi}\int_{0}^{\infty} B^*(\lambda)e^{-\pi\lambda}\Lambda(\lambda-N)d\lambda$$

$$\Lambda(\lambda-N) = (-1)^N(-\frac{4\pi^2\cos(\lambda\pi)}{\lambda-N} - \frac{4\pi(\cos(\pi\lambda)-\sin(\pi\lambda))}{(\lambda-N)^2} + \frac{4\sin(\pi\lambda)}{(\lambda-N)^3})$$

Let

$$\Lambda^*(\lambda, N) = -\frac{4\pi^2\cos(\lambda\pi)}{\lambda-N} - \frac{4\pi(\cos(\pi\lambda)-\sin(\pi\lambda))}{(\lambda-N)^2} + \frac{4\sin(\pi\lambda)}{(\lambda-N)^3}$$

Then we get,

$$R_B(N) = \frac{(-1)^N}{2\pi}\int_{0}^{\infty} B^*(\lambda)e^{-\pi(\lambda-N)}\Lambda^*(\lambda, N)d\lambda$$

Therefore, if we denote by

$$W(N) = \frac{1}{2\pi}\int_{0}^{\infty} B^*(\lambda)e^{-\pi(\lambda-N)}\Lambda^*(\lambda, N)d\lambda$$

we have $R_B(N) = (-1)^N W(N)$. Since, $R_B(N) \geq 0$ for every N, we can see that $W(N) \geq 0$ if N is even and $W(N) \leq 0$ if N is odd.

Proposition 3.1. *There exists a constant $C > 0$ such that for every natural number N, we have*

$$W(N) =$$

$$\frac{1}{2\pi}\int_{(\lambda-N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-N)}|\Lambda^*(\lambda, N)|d\lambda - \frac{1}{2\pi}\int_{(\lambda-N)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-N)}|\Lambda^*(\lambda, N)|d\lambda$$

$$+O(\int_{|\lambda-N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-N)}min(1, \frac{1}{(\lambda-N)^2})d\lambda)$$

Proof. We have,

$$W(N)=\frac{1}{2\pi}\int_0^\infty B^*(\lambda)e^{-\pi(\lambda-N)}\Lambda^*(\lambda,N)d\lambda=$$

$$\frac{1}{2\pi}\int_{(\lambda-N)\cos(\pi\lambda)<0}B^*(\lambda)e^{-\pi(\lambda-N)}\Lambda^*(\lambda,N)d\lambda+\frac{1}{2\pi}\int_{(\lambda-N)\cos(\pi\lambda)>0}B^*(\lambda)e^{-\pi(\lambda-N)}\Lambda^*(\lambda,N)d\lambda=$$

$$\frac{1}{2\pi}\int_{(\lambda-N)\cos(\pi\lambda)<0}B^*(\lambda)e^{-\pi(\lambda-N)}|\Lambda^*(\lambda,N)|d\lambda-\frac{1}{2\pi}\int_{(\lambda-N)\cos(\pi\lambda)>0}B^*(\lambda)e^{-\pi(\lambda-N)}|\Lambda^*(\lambda,N)|d\lambda-$$

$$\frac{1}{\pi}\int_{(\lambda-N)\cos(\pi\lambda)<0}B^*(\lambda)e^{-\pi(\lambda-N)}max(-\Lambda^*(\lambda,N),0)d\lambda+$$

$$\frac{1}{\pi}\int_{(\lambda-N)\cos(\pi\lambda)>0}B^*(\lambda)e^{-\pi(\lambda-N)}max(\Lambda^*(\lambda,N),0)d\lambda$$

We have

$$\Lambda^*(\lambda,N)=-\frac{4\pi^2\cos(\lambda\pi)}{\lambda-N}-\frac{4\pi(\cos(\pi\lambda)-\sin(\pi\lambda))}{(\lambda-N)^2}+\frac{4\sin(\pi\lambda)}{(\lambda-N)^3}$$

Assume that $\Lambda^*(\lambda,N)\leq 0$ and $(\lambda-N)\cos(\pi\lambda)<0$. Then, we get

$$|\lambda-N|4\pi^2|\cos(\pi\lambda)|\leq 4\pi(\cos(\pi\lambda)-\sin(\pi\lambda))-\frac{4\sin(\lambda\pi)}{\lambda-N}$$

Similarly, if $\Lambda^*(\lambda,N)\geq 0$ and $(\lambda-N)\cos(\pi\lambda)>0$, we get that

$$|\lambda-N|4\pi^2|\cos(\pi\lambda)|\leq -4\pi(\cos(\pi\lambda)-\sin(\pi\lambda))+\frac{4\sin(\lambda\pi)}{\lambda-N}$$

Since, the function $4\pi(\cos(\pi\lambda)-\sin(\pi\lambda))$ is bounded and the the function $\frac{4\sin(\lambda\pi)}{\lambda}$ is also bounded and using the fact that

$$\frac{4\sin(\lambda\pi)}{\lambda-N}=(-1)^N\frac{4\sin((\lambda-N)\pi)}{\lambda-N}$$

, we have that there exists a constant $C>0$ independent of N, such that in both cases

$$|\lambda-N||\cos(\pi\lambda)|\leq C$$

Therefore, we have

$$\int_{(\lambda-N)\cos(\pi\lambda)<0}B^*(\lambda)e^{-\pi(\lambda-N)}max(-\Lambda^*(\lambda,N),0)d\lambda=O(\int_{|\lambda-N||\cos(\pi\lambda)|\leq C}B^*(\lambda)e^{-\pi(\lambda-N)}|\Lambda^*(\lambda,N)|d\lambda)$$

and

$$\int_{(\lambda-N)\cos(\pi\lambda)>0}B^*(\lambda)e^{-\pi(\lambda-N)}max(\Lambda^*(\lambda,N),0)d\lambda=O(\int_{|\lambda-N||\cos(\pi\lambda)|\leq C}B^*(\lambda)e^{-\pi(\lambda-N)}|\Lambda^*(\lambda,N)|d\lambda)$$

Now, since $\lim_{\lambda\to 0}\Lambda(\lambda)$ exists and it is a real number and also $\lim_{|\lambda|\to+\infty}\Lambda(\lambda)=0$, we get by continuity, that the function $\Lambda(\lambda)$ is bounded.

Now, using the fact that $\Lambda^*(\lambda,N)=(-1)^N\Lambda(\lambda-N)$, we have that

$$|\Lambda^*(\lambda,N)|=O(1)$$

Also, if we have that $|\lambda-N||\cos(\pi\lambda)|\leq C$, we get that

$$|\Lambda^*(\lambda,N)|=O(\frac{1}{(\lambda-N)^2}+\frac{1}{|\lambda-N|^3})$$

Therefore, we get that for λ such that $|\lambda-N||\cos(\pi\lambda)|\leq C$, we have

$$|\Lambda^*(\lambda,N)|=O(min(1,\frac{1}{(\lambda-N)^2}))$$

together with an absolute implied constant.

Therefore, we get

$$\int_{(\lambda-N)\cos(\pi\lambda)<0}B^*(\lambda)e^{-\pi(\lambda-N)}max(-\Lambda^*(\lambda,N),0)d\lambda=$$

$$O(\int_{|\lambda-N||\cos(\pi\lambda)|\leq C}B^*(\lambda)e^{-\pi(\lambda-N)}min(1,\frac{1}{(\lambda-N)^2})d\lambda)$$

and

$$\int_{(\lambda-N)\cos(\pi\lambda)>0}B^*(\lambda)e^{-\pi(\lambda-N)}max(\Lambda^*(\lambda,N),0)d\lambda=$$

$$O(\int_{|\lambda-N||\cos(\pi\lambda)|\leq C}B^*(\lambda)e^{-\pi(\lambda-N)}min(1,\frac{1}{(\lambda-N)^2})d\lambda)$$

This completes the proof. □

We continue with the following Proposition.

Proposition 3.2. *We have,*

$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0}B^*(\lambda)e^{-\pi(\lambda-2N)}\Lambda^*(\lambda,2N+1)d\lambda-$$

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0}B^*(\lambda)e^{-\pi(\lambda-2N)}\Lambda^*(\lambda,2N)d\lambda=$$

$$-\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0}B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)||d\lambda+$$

$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)} min(1, \frac{1}{(\lambda-2N)^2}d\lambda)$$
$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

Proof. By Proposition 3.1, we get

$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}\Lambda^*(\lambda,2N+1)d\lambda -$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}\Lambda^*(\lambda,2N)d\lambda =$$
$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda -$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda +$$
$$+O(\int_{|\lambda-2N-1||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)} min(1, \frac{1}{(\lambda-2N-1)^2}d\lambda)+$$
$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)} min(1, \frac{1}{(\lambda-2N)^2}d\lambda)$$

We have,

$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda -$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda =$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)| - |\Lambda^*(\lambda,2N)|d\lambda +$$
$$O(\int_{2N}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

This is true because, if $\cos(\pi\lambda) < 0$, then forces $\lambda > 2N$ in both of the terms and also if $(\lambda - 2N)\cos(\pi\lambda) \geq 0$ in the first term and $\cos(\pi\lambda) > 0$, then surely $\lambda \geq 2N$.

Now,

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)| - |\Lambda^*(\lambda,2N)|d\lambda =$$
$$-\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)| - |\Lambda^*(\lambda,2N)||d\lambda +$$

$$\frac{1}{\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}max(|\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)|,0)d\lambda =$$

$$-\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)||d\lambda +$$

$$\frac{1}{\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0,|\Lambda(\lambda,2N+1)|\geq|\Lambda(\lambda,2N)|} B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)||d\lambda =$$

$$-\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)||d\lambda +$$

$$\frac{1}{\pi}\int_{\lambda<2N-1,\cos(\pi\lambda)>0,\Lambda(\lambda,2N-1)\geq\Lambda(\lambda,2N)\geq 0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)-\Lambda^*(\lambda,2N)|d\lambda +$$

$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)$$

$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

But if we have $\lambda < 2N-1$ and $\cos(\pi\lambda) > 0$ and $\Lambda(\lambda,2N+1) \geq \Lambda(\lambda,2N)$, then

$$-\frac{4\pi^2\cos(\lambda\pi)}{\lambda-2N-1} - \frac{4\pi(\cos(\pi\lambda)-\sin(\pi\lambda))}{(\lambda-2N-1)^2} + \frac{4\sin(\pi\lambda)}{(\lambda-2N-1)^3} \geq$$

$$-\frac{4\pi^2\cos(\lambda\pi)}{\lambda-2N} - \frac{4\pi(\cos(\pi\lambda)-\sin(\pi\lambda))}{(\lambda-2N)^2} + \frac{4\sin(\pi\lambda)}{(\lambda-2N)^3}$$

Therefore,

$$-4\pi|\cos(\lambda\pi)|(\frac{\pi}{\lambda-2N-1} - \frac{\pi}{\lambda-2N} + \frac{1}{(\lambda-2N-1)^2} - \frac{1}{(\lambda-2N)^2}) \geq$$

$$-4\pi\sin(\lambda\pi)(\frac{1}{(\lambda-2N-1)^2} - \frac{1}{(\lambda-2N)^2} + \frac{1}{\pi(\lambda-2N-1)^3} - \frac{1}{(\lambda-2N)^3})$$

which, gives that

$$|\cos(\lambda\pi)|((\frac{\pi}{(\lambda-2N-1)(\lambda-2N)} + \frac{2\lambda-4N-1}{(\lambda-2N-1)^2(\lambda-2N)^2}) \leq$$

$$\sin(\lambda\pi)(\frac{2\lambda-4N-1}{(\lambda-2N-1)^2(\lambda-2N)^2} + \frac{3(\lambda-2N-1)^2+3(\lambda-2N-1)+1}{\pi(\lambda-2N-1)^3(\lambda-2N)^3})$$

Therefore, we get

$$|\cos(\lambda\pi)||\lambda-2N-1||\lambda-2N| \leq C_1(2\lambda-4N-1) + C_2\frac{3(\lambda-2N-1)^2+3(\lambda-2N-1)+1}{(\lambda-2N-1)(\lambda-2N)}$$

, which gives that

$$|\cos(\lambda\pi)||\lambda-2N| \leq C_1\frac{2\lambda-4N-1}{\lambda-2N-1}+C_2\frac{3}{\lambda-2N}+C_2\frac{3}{(\lambda-2N)(\lambda-2N-1)}+C_2\frac{1}{(\lambda-2N)(\lambda-2N-1)^2}$$

Using the fact that $\lambda < 2N-1$, we can easily see that there exists $C > 0$ independent of N and λ such that

$$|\cos(\lambda\pi)||\lambda-2N| \leq C$$

Therefore, we can easily see that

$$\frac{1}{\pi}\int_{\lambda<2N-1,\cos(\pi\lambda)>0,\Lambda(\lambda,2N-1)\geq\Lambda(\lambda,2N)\geq 0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)-\Lambda^*(\lambda,2N)|d\lambda =$$

$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

This completes the proof. □

Proposition 3.3. *We have that there exists $C > 0$ such that for every natural number N*

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)||d\lambda \leq$$

$$W(2N)-e^{\pi}W(2N+1)+O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda+\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

Proof. By Proposition 3.2, we get

$$-\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)||d\lambda =$$

$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}\Lambda^*(\lambda,2N+1)d\lambda-$$

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}\Lambda^*(\lambda,2N)d\lambda+$$

$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)$$

$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

Also, we have

$$e^{\pi}W(2N+1)-W(2N) =$$

$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda-$$

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda-$$
$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda+$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda+$$
$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)+$$
$$O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

Therefore, we get

$$-\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)||d\lambda=$$
$$e^{\pi}W(2N+1)-W(2N)+\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda-$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)d\lambda|+$$
$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)$$
$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

It suffices to show that

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda-$$
$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda\geq$$
$$2e^{\pi}W(2N+1)-2W(2N)$$
$$+O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)$$
$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

,which is true if and only if

$$\frac{1}{\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda-$$

$$\frac{1}{\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda-$$
$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda+$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda \leq$$
$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)+$$
$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

, which is true if and only if

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda-$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda \geq$$
$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda-$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda+$$
$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda-$$
$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N+1)|d\lambda+$$
$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)+$$
$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

This always holds, firstly because $W(2N)\geq 0$, therefore always

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda-$$
$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda,2N)|d\lambda \geq$$
$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)+$$

$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

Secondly, we have $e^{\pi}W(2N+1) \leq 0$. Therefore

$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda, 2N+1)|d\lambda-$$

$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda, 2N+1)|d\lambda \leq$$

$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1, \frac{1}{(\lambda-2N)^2})d\lambda)+$$

$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

and finally because as we showed in the previous Proposition we have

$$\frac{1}{2\pi}\int_{(\lambda-2N-1)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda, 2N+1)|d\lambda-$$

$$\frac{1}{2\pi}\int_{(\lambda-2N)\cos(\pi\lambda)<0} B^*(\lambda)e^{-\pi(\lambda-2N)}|\Lambda^*(\lambda, 2N)|d\lambda \leq$$

$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1, \frac{1}{(\lambda-2N)^2})d\lambda)$$

$$+O(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

This completes the proof. □

4. Proof of the Erdos-Turan Conjecture

In this section we will present a full proof of the Erdos-Turin Conjecture on Additive Bases.

This conjecture states that if for a set B of natural numbers we have that $R_B(N) > 0$ for N sufficiently large, then we must have that $\limsup_{N\to\infty} R_B(N) = \infty$.

From now on we will write

$$L(\lambda) = |\Lambda(\lambda-1)| - |\Lambda(\lambda)|$$

Now, for every asymptotic additive base of the positive integrers B, we have $B(\lambda) \geq c\sqrt{\lambda}$ for every $\lambda > M_0$, where M_0 is the least element of B, where $c > 0$ is a constant, by [4]. Also, for every B with $R_B(N) = O(1)$, we get that

$$\{(a,b) \in B^2 : a, b \leq \lambda\} \subseteq \cup_{k \leq 2\lambda, k \in \mathbb{N}} \{(a,b) \in B^2 : a + b = k\}$$

,which gives

$$B(\lambda)^2 \leq \sum_{k \leq 2\lambda} R_B(k) \leq v\lambda$$

for some constant $v > 0$. This shows that

$$B(\lambda) \leq c\sqrt{\lambda}$$

for some constant $c > 0$.

Therefore, by Lemma 2.3, we get $w_1\lambda^2 \leq B^*(\lambda) \leq w_2\lambda^2$, for some constants $w_1, w_2 > 0$ and every $\lambda > 2M_0$.

Proposition 4.1. *Assume that for a set B of natural numbers we have*

$$\frac{N^2 \int_{\lambda < 2N, \cos(\pi\lambda) > 0} B^*(\lambda) e^{-\pi\lambda} |L(\lambda - 2N)| d\lambda}{e^{\pi N} (\int_{|\lambda - 2N||\cos(\pi\lambda)| \leq C} B^*(\lambda) min(1, \frac{1}{(\lambda-2N)^2}) e^{-\pi\lambda} d\lambda)(\int_{2N-1}^{\infty} B^*(\lambda) e^{-\frac{\pi}{2}\lambda} d\lambda)} \to \infty$$

as $N \to \infty$. Then, $\limsup_{N \to \infty} R_B(N) = \infty$.

Proof. Assume for the sake of contradiction that $R_B(N) = O(1)$.

Therefore by Proposition 3.3, we get that

$$\frac{1}{2\pi} \int_{(\lambda - 2N)\cos(\pi\lambda) < 0, \cos(\pi\lambda) > 0} B^*(\lambda) e^{-\pi(\lambda - 2N)} ||\Lambda^*(\lambda, 2N+1)| - |\Lambda^*(\lambda, 2N)|| d\lambda \leq$$

$$W(2N) - e^{\pi} W(2N+1) +$$

$$O(\int_{|\lambda - 2N||\cos(\pi\lambda)| \leq C} B^*(\lambda) e^{-\pi(\lambda - 2N)} min(1, \frac{1}{(\lambda - 2N)^2}) d\lambda + \int_{2N-1}^{\infty} B^*(\lambda) e^{-\pi(\lambda - 2N)} d\lambda)$$

Therefore, we get that

$$\frac{1}{2\pi} \int_{(\lambda - 2N)\cos(\pi\lambda) < 0, \cos(\pi\lambda) > 0} B^*(\lambda) e^{-\pi(\lambda - 2N)} ||\Lambda^*(\lambda, 2N+1)| - |\Lambda^*(\lambda, 2N)|| d\lambda \leq$$

$$W(2N) - e^{\pi} W(2N-1) +$$

$$O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda+\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

By assumption, we get

$$W(N)-e^{\pi}W(2N-1)=O(1)$$

Therefore, we can write the following

$$\int_{(\lambda-2N)\cos(\pi\lambda)<0,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi(\lambda-2N)}||\Lambda^*(\lambda,2N+1)|-|\Lambda^*(\lambda,2N)||d\lambda\leq$$

$$O(1)+O(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi(\lambda-2N)}min(1,\frac{1}{(\lambda-2N)^2})d\lambda+\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi(\lambda-2N)}d\lambda)$$

Changing variables we have that

$$0\leq\int_{\lambda<0,\cos(\pi\lambda)>0} B^*(\lambda+2N)e^{-\pi\lambda}|L(\lambda)|d\lambda\leq$$

$$O(1)+O(\int_{|\lambda||\cos(\pi\lambda)|\leq C} B^*(\lambda+2N)e^{-\pi\lambda}min(1,\frac{1}{\lambda^2})d\lambda+\int_{-1}^{\infty} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda)$$

The inequality $B^*(\lambda+2N)\geq(\lambda+2N)^2\chi_{(2M_0,\infty)}(\lambda+2N)$ also shows that

$$\int_{-1}^{\infty} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda\geq uN^2$$

and

$$\int_{|\lambda||\cos(\pi\lambda)|\leq C} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda\geq u^{'}N^2$$

for some constants $u,u^{'}>0$ independent of N and N sufficiently large.

Therefore, dividing this asymptotic double inequality by

$$(\int_{|\lambda||\cos(\pi\lambda)|\leq C} B^*(\lambda+2N)e^{-\pi\lambda}min(1,\frac{1}{\lambda^2})d\lambda)(\int_{-1}^{\infty} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda)\frac{1}{N^2}$$

we get

$$\frac{N^2\int_{\lambda<0,\cos(\pi\lambda)>0} B^*(\lambda+2N)e^{-\pi\lambda}|L(\lambda)|d\lambda}{(\int_{|\lambda||\cos(\pi\lambda)|\leq C} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda)(\int_{-1}^{\infty} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda)}$$

$$=O(1)$$

Also, we have

$$(\int_{|\lambda||\cos(\pi\lambda)|\leq C} B^*(\lambda+2N)e^{-\pi\lambda}min(1,\frac{1}{\lambda^2})d\lambda)(\int_{-1}^{\infty} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda)=$$

$$e^{4\pi N}(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\pi\lambda}d\lambda) \leq$$

$$C^{'}e^{3\pi N}(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C}(B^*(\lambda))e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\frac{\pi}{2}\lambda}d\lambda)$$

for some constant $C^{'} > 0$. We have

$$\int_{\lambda<2N,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi\lambda}|L(\lambda-2N)|d\lambda = e^{-2\pi N}\int_{\lambda<0,\cos(\pi\lambda)>0} B^*(\lambda+2N)e^{-\pi\lambda}|L(\lambda)|$$

By assumption we have that for every $W > 0$

$$\int_{\lambda<2N,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi\lambda}|L(\lambda-2N)|d\lambda$$

$$\geq W\frac{e^{\pi N}}{N^2}(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\frac{\pi}{2}\lambda}d\lambda)$$

for N large enough.

Therefore, we get that for every $W > 0$

$$\int_{\lambda<0,\cos(\pi\lambda)>0} B^*(\lambda+2N)e^{-\pi\lambda}|L(\lambda)| \geq \frac{W}{N^2}(\int_{|\lambda||\cos(\pi\lambda)|\leq C} B^*(\lambda+2N)e^{-\pi\lambda}min(1,\frac{1}{\lambda^2})d\lambda)(\int_{-1}^{\infty} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda)$$

for N large enough. This contradicts the fact that

$$\frac{N^2\int_{\lambda<0,\cos(\pi\lambda)>0} B^*(\lambda+2N)e^{-\pi\lambda}|L(\lambda)|d\lambda}{(\int_{|\lambda||\cos(\pi\lambda)|\leq C} B^*(\lambda+2N)e^{-\pi\lambda}min(1,\frac{1}{\lambda^2})d\lambda)(\int_{-1}^{\infty} B^*(\lambda+2N)e^{-\pi\lambda}d\lambda)}$$

$$= O(1)$$

This completes the proof. □

Theorem 4.2. *Erdos Turan conjecture is true.*

Proof. We have by Fatou's Lemma

$$\liminf_{N\to\infty}\int_{\lambda<2N,\cos(\pi\lambda)>0} N^2B^*(\lambda)e^{-\pi\lambda}|L(\lambda-2N)|d\lambda \geq$$

$$\int_{\cos(\pi\lambda)>0}\liminf_{N\to\infty}(B^*(\lambda)N^2|L(\lambda-2N)|e^{-\pi\lambda}\chi_{(M_0,2N)}(\lambda))d\lambda$$

Also, by a computation on Mathematica we get

$$\lim_{N\to\infty} N^2|L(\lambda-N)| = \frac{\pi^2}{\sqrt{2}}\sqrt{1+\cos(2\pi\lambda)}$$

Therefore, it is easy to see that

$$\int_{\lambda<2N,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi\lambda}|L(\lambda-2N)|d\lambda \geq \frac{w}{N^2}$$

for some constant $w>0$ independent of N. Then, we have

$$\frac{N^2\int_{\lambda<2N,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi\lambda}|L(\lambda-2N)|d\lambda}{e^{\pi N}(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\frac{\pi}{2}\lambda}d\lambda)} \geq$$

$$\frac{w}{e^{\pi N}(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\frac{\pi}{2}\lambda}d\lambda)}$$

Since $B^*(\lambda)\leq C\lambda^2$, we get

$$\int_{2N-1}^{\infty} B^*(\lambda)e^{-\frac{\pi}{2}\lambda}d\lambda \leq C\int_{2N-1}^{\infty}\lambda^2 e^{-\frac{\pi}{2}\lambda}d\lambda = O(N^2e^{-\pi N})$$

Therefore, we get

$$\frac{N^2\int_{\lambda<2N,\cos(\pi\lambda)>0} B^*(\lambda)e^{-\pi\lambda}|L(\lambda-2N)|d\lambda}{e^{\pi N}(\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda)(\int_{2N-1}^{\infty} B^*(\lambda)e^{-\frac{\pi}{2}\lambda}d\lambda)} \geq$$

$$\frac{w}{N^2\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} B^*(\lambda)e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda} \geq$$

$$\frac{w}{N^2\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} \lambda^2e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda}$$

Surely, we have

$$N^2\int_N^{\infty}\lambda^2e^{-\pi\lambda}d\lambda\to 0$$

as $N\to\infty$. Now if $\lambda\in(1,N)$, then $|\lambda-2N|\geq N$ Therefore, we can write

$$\frac{w}{N^2\int_{|\lambda-2N||\cos(\pi\lambda)|\leq C} \lambda^2e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda} \geq$$

$$\frac{w}{N^2\int_{|\cos(\pi\lambda)|\leq\frac{C}{N},1<\lambda<N} \lambda^2e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda+o(1)}$$

To prove the Theorem it suffices to show that

$$N^2\int_{|\cos(\pi\lambda)|\leq\frac{C}{N},1<\lambda<N} \lambda^2e^{-\pi\lambda}min(1,\frac{1}{(\lambda-2N)^2})d\lambda=o(1)$$

Since, $\lambda \in (1, N)$, then $|\lambda - 2N| \geq N$.

Therefore,

$$min(1, \frac{1}{(\lambda - 2N)^2}) \leq \frac{1}{N^2}$$

Therefore,

$$N^2 \int_{|\cos(\pi\lambda)| \leq \frac{C}{N}, 1<\lambda<N} \lambda^2 e^{-\pi\lambda} min(1, \frac{1}{(\lambda - 2N)^2}) d\lambda \leq$$
$$\int_{|\cos(\pi\lambda)| \leq \frac{C}{N}, 1<\lambda<N} \lambda^2 e^{-\pi\lambda} d\lambda = \int_1^{\infty} \lambda^2 e^{-\pi\lambda} \chi_{A_N}(\lambda) d\lambda$$

Here $A_n = \{\lambda \in (1, N) : |\cos(\pi\lambda)| \leq \frac{C}{N}\}$. Obviously, we have that

$$\lim_{N \to \infty} \chi_{A_N}(\lambda) = 0$$

for every $\lambda \in (1, \infty)$ such that $\cos(\pi\lambda) \neq 0$.

Additionally the Lebesgue measure of the set $\{\lambda \in (1, \infty) : \cos(\pi\lambda) = 0\}$ is equal to 0.

Thus, by dominated convergence theorem, we conclude that

$$\int_{|\cos(\pi\lambda)| \leq \frac{C}{N}, 1<\lambda<N} \lambda^2 e^{-\pi\lambda} d\lambda = o(1)$$

This completes the proof. □

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