Bucharest, March 4th, 2011

SOUTH EASTERN EUROPEAN MATHEMATICAL OLYMPIAD FOR UNIVERSITY STUDENTS
Bucharest, March 4th, 2011

Solutions and grading schemes

Problem 1. For a given integer $n \geq 1$, let $f : [0, 1] \to \mathbb{R}$ be a non-decreasing function. Prove that

$$\int_0^1 f(x)\ dx \leq (n+1) \int_0^1 x^n f(x) \mathrm{d}x.$$

Find all non-decreasing continuous functions for which equality holds.

Solution. For given n and any $x, y \in [0, 1]$ we integrate on the interval $[0, 1]$ in terms of x and in terms of y the obvious inequality

$$(x^n - y^n)(f(x) - f(y)) \geq 0, \tag{1}$$

obtaining

$$\int_0^1 \left(\int_0^1 (x^n - y^n)(f(x) - f(y))\ \mathrm{d}x \right)\ \mathrm{d}y \geq 0,$$

or

$$\int_0^1 x^n f(x)\ \mathrm{d}x - \int_0^1 y^n\ \mathrm{d}y \int_0^1 f(x)\ \mathrm{d}x - \int_0^1 x^n\ \mathrm{d}x \int_0^1 f(y)\ \mathrm{d}y + \int_0^1 y^n f(y)\ \mathrm{d}y \geq 0$$

which gives the desired inequality. As f is continuous, when equality holds it forces equality in (1) also, that is f must be constant.

Grading scheme.

$(n+1) \int_0^1 x^n f(x) \mathrm{d}x = \int_0^1 f\left(\sqrt[n+1]{t}\right) \mathrm{d}t$ 3p

f non-decreasing implies $f(x) \leq f\left(\sqrt[n+1]{x}\right)$ for $x \in [0, 1]$ 3p

$\int\limits_0^1 f(x)\,\mathrm{d}x \leq \int\limits_0^1 f(\sqrt[n+1]{x})\,\mathrm{d}x = (n+1)\int_0^1 x^n f(x)\,\mathrm{d}x$ 1p

Equality holds only for constant functions. 3p

Alternative solution.

$(x^n - y^n)(f(x) - f(y)) \geq 0$ 3p

Integrate the above inequality on $[0,1]\times[0,1]$ to obtain

$0 \leq \int\limits_0^1 \int\limits_0^1 ((x^n - y^n)(f(x) - f(y)))\,\mathrm{d}x\mathrm{d}y = \int\limits_0^1 x^n f(x)\,\mathrm{d}x - \int\limits_0^1 x^n \mathrm{d}x \int\limits_0^1 f(y)\,\mathrm{d}y - \int\limits_0^1 y^n \mathrm{d}y \int\limits_0^1 f(x)\,\mathrm{d}x + \int\limits_0^1 y^n f(y)\,\mathrm{d}y$ 3p

Using $\int\limits_0^1 x^n \mathrm{d}x = \dfrac{1}{n+1}$ it obtains the inequality. 1p

Equality holds only for constants function. 3p

Remark 1 *The use of the Chebyshev inequality (without proof)* *7p*

Problem 2. Let $A = (a_{ij})$ be a real $n \times n$ matrix such that $A^n \neq 0$ and $a_{ij}a_{ji} \leq 0$, for all i, j. Prove that there exist two nonreal numbers among eigenvalues of A.

Solution. Let $\lambda_1, \ldots, \lambda_n$ be all eigenvalues of this matrix. From condition it follows that $a_{ii} = 0$, therefore $\sum\limits_{k=1}^{n} \lambda_k = 0$. The sum $\sum\limits_{i<j} \lambda_i \lambda_j$ equals the sum of principal minors with size 2 for the matrix A, i. e. it equals $-\sum\limits_{i<j} a_{ij}a_{ji}$. Thus the sum $\sum\limits_{k=1}^{n} \lambda_k^2 = 2\sum\limits_{i<j} a_{ij}a_{ji} \leq 0$. If a number z is an eigenvalue of A then so does $\overline{z}$.

Grading scheme. $a_{ii} = 0$ 1p

$\sum \lambda_i$ 1p

$\sum\limits_{i<j} \lambda_i \lambda_j \geq 0$ 3p

$\sum\limits_{i=1}^{n} \lambda_i^2 \leq 0$ 2p

$A^n \neq 0$ implies $\exists \lambda_i \neq 0$ 1p

$\sum \lambda_i^2 < 0$ implies $\exists \lambda_i \in \mathbb{C}\backslash\mathbb{R}$ 1p

$\exists \lambda_i \in \mathbb{C}\backslash\mathbb{R}$ implies $\exists \lambda_j = \overline{\lambda_i} \neq \lambda_i \in \mathbb{C}\backslash\mathbb{R}$ 1p

Alternative solution. $\mathrm{Tr}(A^2) \leq 0$ 4p

$\mathrm{Tr}(A^2) = \sum \lambda_i^2$ 1p

$\sum \lambda_i^2 \leq 0$ 2p

$A^n \neq 0$ implies $\exists \lambda_i \neq 0$ 1p
$\sum \lambda_i^2 < 0$ implies $\exists \lambda_i \in \mathbb{C} \backslash \mathbb{R}$ 1p
$\exists \lambda_i \in \mathbb{C} \backslash \mathbb{R}$ implies $\exists \lambda_j = \overline{\lambda_i} \neq \lambda_i \in \mathbb{C} \backslash \mathbb{R}$ 1p

Problem 3. Given vectors $\overline{a}, \overline{b}, \overline{c} \in \mathbb{R}^n$, show that

$$\left(||\overline{a}|| \left\langle \overline{b}, \overline{c} \right\rangle\right)^2 + \left(||\overline{b}|| \left\langle \overline{a}, \overline{c} \right\rangle\right)^2 \leq ||\overline{a}|| ||\overline{b}|| \left(||\overline{a}|| ||\overline{b}|| + |\left\langle \overline{a}, \overline{b} \right\rangle|\right) ||\overline{c}||^2,$$

where $\langle \overline{x}, \overline{y} \rangle$ denotes the scalar (inner) product of the vectors $\overline{x}$ and $\overline{y}$ and $||\overline{x}||^2 = \langle \overline{x}, \overline{x} \rangle$.

Solution (Trigonometric). We shall first prove that, given vectors $u, v \in \mathbb{R}^n$, with $||u|| = ||v|| = 1$ (thus $u, v \in S^{n-1}$), then

$$\sup_{||x||=1} \left(\langle u, x \rangle^2 + \langle v, x \rangle^2\right) = 1 + |\langle u, v \rangle|.$$

Consider the unique vectorial representation $x = x' + x''$ with $x' \in \operatorname{span}(u, v)$ and $x'' \perp \operatorname{span}(u, v)$. Then $1 = ||x||^2 = \langle x' + x'', x' + x'' \rangle = ||x'||^2 + 2\langle x', x'' \rangle + ||x''||^2 = ||x'||^2 + ||x''||^2$ (Pythagoras' relation), so $||x'|| \leq 1$. Now, $|\langle u, x \rangle| = |\langle u, x' \rangle| \leq |\langle u, y \rangle|$ where $y = 0$ if $x' = 0$ and $y = \dfrac{x'}{||x'||}$ if $x' \neq 0$ (thus $||y|| = 1$). Similarly $|\langle v, x \rangle| = |\langle v, x' \rangle| \leq |\langle v, y \rangle|$. The maximum asked is therefore obtained when $x \in \operatorname{span}(u, v)$.

Thus our vectorial problem has been transferred in the 2-dimensional space, with unit vectors u, v and x. The endpoints of the vectors $\pm u$ and $\pm v$ partition the circle S^1 into four arcs, each of measure at most π (and possibly 0, when $u = \pm v$); the endpoint of x will fall into one of them. Let ω be the measure of that arc, and α, β, with $\alpha + \beta = \omega$, the measures of the arcs between the endpoint of x and the ends of that arc. Then $\langle u, x \rangle^2 + \langle v, x \rangle^2 = \cos^2 \alpha + \cos^2 \beta$, by the well-known geometric interpretation of the dot-product (independently of the dimension of the space). But then $\langle u, x \rangle^2 + \langle v, x \rangle^2 = \cos^2 \alpha + \cos^2 \beta = 1 + \dfrac{1}{2}(\cos 2\alpha + \cos 2\beta) = 1 + \cos(\alpha + \beta)\cos(\alpha - \beta) \leq 1 + |\cos \omega| = 1 + |\langle u, v \rangle|$, with equality in the obvious cases

- when $\alpha = \beta = \omega/2$, therefore when $|\langle u, x \rangle| = |\langle v, x \rangle|$, therefore $x = (\pm u \pm v)/|| \pm u \pm v||$, where the signs are such that $0 \leq \omega < \pi/2$;
- when $\omega = \pi/2$, therefore when $\langle u, v \rangle = 0$, i.e. $u \perp v$, for all $x \in \operatorname{span}(u, v)$.

Then, for any not-null a, b, c, let us take $u = \dfrac{a}{||a||}$, $v = \dfrac{b}{||b||}$, $x = \dfrac{c}{||c||}$, therefore $\langle u, x \rangle^2 = \dfrac{1}{||a|| \cdot ||c||^2} \langle a, c \rangle^2$, $\langle v, x \rangle^2 = \dfrac{1}{||b|| \cdot ||c||^2} \langle b, c \rangle^2$, $\langle u, v \rangle =$

$\dfrac{\langle a, b\rangle}{||a|| \cdot ||b||}$, so the proven relation becomes

$$\frac{1}{||a||} \langle a, c\rangle^2 + \frac{1}{||b||} \langle b, c\rangle^2 \leq \left(1 + \frac{|\langle a, b\rangle|}{||a|| \cdot ||b||}\right) ||c||^2,$$

equivalent with the required inequality, also true for a, b or c equal to zero.

Remarks. We can continue by AM-GM to obtain

$$|\langle u, x\rangle \cdot \langle v, x\rangle| \leq \frac{1}{2}\left(\langle u, x\rangle^2 + \langle v, x\rangle^2\right) \leq \frac{1}{2}\left(1 + |\langle u, v\rangle|\right).$$

A simple computation now yields

$$|\langle a, c\rangle \cdot \langle b, c\rangle| \leq \frac{1}{2}(||a|| \cdot ||b|| + |\langle a, b\rangle|)||c||^2,$$

also true for a, b or c equal to zero, a generalization of the Cauchy-Schwarz inequality, since for $a = b$ it precisely yields it.

Solution (Quadratic Forms). For $||x|| = ||u|| = ||v|| = 1$ we have

$$\begin{aligned} 0 \leq ||\lambda x + \mu u + \nu v||^2 &= \langle \lambda x + \mu u + \nu v, \lambda x + \mu u + \nu v\rangle = \\ &= \lambda^2 + \mu^2 + \nu^2 + 2\lambda\mu \langle x, u\rangle + 2\lambda\nu \langle x, v\rangle + 2\mu\nu \langle u, v\rangle, \end{aligned}$$

a quadratic form which takes non-negative values for any real parameters λ, μ, ν, thus corresponding to a positive semidefined matrix

$$\begin{bmatrix} 1 & \langle x, u\rangle & \langle x, v\rangle \\ \langle x, u\rangle & 1 & \langle u, v\rangle \\ \langle x, v\rangle & \langle u, v\rangle & 1 \end{bmatrix}$$

The principal minors of order 1 are thus non-negative, which yields the semi-positivity of the norm; the principal minors of order 2 are non-negative, which yields the C-B-S inequality, e.g. $1 \geq \langle u, v\rangle^2$; finally, also the determinant of the matrix is non-negative

$$\Delta = 1 - (\langle u, v\rangle^2 + \langle x, u\rangle^2 + \langle x, v\rangle^2) + 2\langle u, v\rangle \langle x, u\rangle \langle x, v\rangle \geq 0,$$

which can be arranged as $\langle x, u\rangle^2 + \langle x, v\rangle^2 \leq 1 + |\langle u, v\rangle| - |\langle u, v\rangle|(1 + |\langle u, v\rangle| - 2|\langle x, u\rangle \langle x, v\rangle|)$. But $\langle x, u\rangle^2 + \langle x, v\rangle^2 \geq 2|\langle x, u\rangle \langle x, v\rangle|$, which plugged into the relation yields that $(1 - |\langle u, v\rangle|)(1 + |\langle u, v\rangle| - 2|\langle x, u\rangle \langle x, v\rangle|) \geq 0$. Then either $1 = |\langle u, v\rangle|$, when $1 + |\langle u, v\rangle| = 2 \geq 2|\langle x, u\rangle \langle x, v\rangle|$ (by C-B-S), or $1 > |\langle u, v\rangle|$, which implies $1 + |\langle u, v\rangle| \geq 2|\langle x, u\rangle \langle x, v\rangle|$. Thus always $\langle x, u\rangle^2 + \langle x, v\rangle^2 \leq 1 + |\langle u, v\rangle|$.

Solution (Lagrange Multipliers). Define

$$L(x, \lambda) = \langle u, x\rangle^2 + \langle v, x\rangle^2 - \lambda(||x||^2 - 1)$$

and consider the system

$$\frac{\partial L}{\partial x_i} = 2u_i \langle u, x\rangle + 2v_i \langle v, x\rangle - 2\lambda x_i = 0,$$

for $1 \leq i \leq n$, and

$$\frac{\partial L}{\partial \lambda} = ||x||^2 - 1 = 0. \text{ Now}$$

$0 = \frac{1}{2}\sum_{i=1}^{n} x_i \frac{\partial L}{\partial x_i} = \langle u, x\rangle \sum_{i=1}^{n} u_i x_i + \langle v, x\rangle \sum_{i=1}^{n} v_i x_i - \lambda \sum_{i=1}^{n} x_i^2 = \langle u, x\rangle^2 + \langle v, x\rangle^2 - \lambda ||x||^2 = \langle u, x\rangle^2 + \langle v, x\rangle^2 - \lambda$, so λ needs be, computed at the critical point(s), the very expression we are considering (of no relevance, in fact). On the other hand, $0 = \frac{1}{2}\sum_{i=1}^{n} u_i \frac{\partial L}{\partial x_i} = \langle u, x\rangle \sum_{i=1}^{n} u_i^2 + \langle v, x\rangle \sum_{i=1}^{n} v_i u_i - \lambda \sum_{i=1}^{n} x_i u_i = \langle u, x\rangle ||u||^2 + \langle v, x\rangle \langle u, v\rangle - \lambda \langle u, x\rangle = \langle u, x\rangle + \langle u, v\rangle \langle v, x\rangle - \lambda \langle u, x\rangle$, and similarly for v, thus reaching the system of two equations (in the variables $\langle u, x\rangle$ and $\langle v, x\rangle$)

$$\begin{cases} (1 - \lambda) \langle u, x\rangle + \langle u, v\rangle \langle v, x\rangle = 0 \\ \langle u, v\rangle \langle u, x\rangle + (1 - \lambda) \langle v, x\rangle = 0 \end{cases}$$

The determinant Δ of the matrix of this particular system is $\Delta = (1-\lambda)^2 - \langle u, v\rangle^2$, and, if not null, the only solution is the trivial $\langle u, x\rangle = \langle v, x\rangle = 0$, when our expression reaches an evident minimum, equal to zero (therefore since then $\lambda = 0$, this means $\langle u, v\rangle \neq \pm 1$, which translates in $u \neq \pm v$, and $x \perp \operatorname{span}(u, v)$). We are therefore interested in $\Delta = 0$ (for all other critical points), leading to $\lambda = 1 \pm \langle u, v\rangle$, thus $\lambda = 1 + | \langle u, v\rangle |$ at a maximum point, and $\lambda = 1 - | \langle u, v\rangle |$ at a critical point. There are some particular cases.

When $u = \pm v$, thus $\langle u, v\rangle = \pm 1$, the situation is simple. We have a maximum of 2 when $x = \pm u$, and a minimum of 0 when $x \perp u$.

When $u \perp v$, thus $\langle u, v\rangle = 0$, then λ (thus the expression) is 1 at the maximum points, for all $x \in \operatorname{span}(u, v)$, and λ (thus the expression) is 0 at the minimum points, for all $x \perp \operatorname{span}(u, v)$.

Grading scheme. **Any solution.** $\|a\| = \|b\| = \|c\| = 1$ to get $\langle a, c\rangle^2 + \langle b, c\rangle^2 \leq 1 + |\langle a, b\rangle|$ 2p

Any solution. Case of some norm is 0 forgotten .... −0p

Any solution. only case of norm 0 .... 1p

Solution 1. (**trigonometry**)

Replacing $\langle\cdot,\cdot\rangle$ with cos to get $\cos^2\alpha+\cos^2\beta\le 1+|\cos\gamma|$ 1p
Replacing $|\cos\gamma|$ with $\cos^2\gamma$ .... +0p, wrong inequality
Some similar cases are lost .... upto -2p 10p

Solution 2. (**coordinates**)

Consider $c=(1,0,\dots,0), a=(a_1,a_2,0,\dots,0), b=(b_1,b_2,\dots,b_n)$ to get $a_1^2+b_1^2\le 1+|a_1b_1+a_2b_2|$ 3p

Moving and adjusting variables $0\le y=b_2\le x=a_1\le 1$ to get $x^2\le y^2+x\sqrt{1-y^2}-y\sqrt{1-x^2}$ 5p

Solution 3. (**Sylvester criteria**)

Consider inequality $\|\lambda a+\mu b+\nu c\|^2\ge 0$ 3p
Witing down Sylvester criteria for form $\omega(\lambda,\mu,\nu)$ 5p

Problem 4. Let $f:[0,1]\to\mathbb{R}$ be a twice continuously differentiable increasing function. Define the sequences given by $L_n=\frac{1}{n}\sum\limits_{k=0}^{n-1} f\left(\frac{k}{n}\right)$ and $U_n=\frac{1}{n}\sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right)$, $n\ge 1$. The interval $[L_n,U_n]$ is divided into three equal segments. Prove that, for large enough n, the number $I=\int\limits_0^1 f(x)\,\mathrm{d}x$ belongs to the middle one of these three segments.

Solution. Let us find a suitable representation for L_n.

Lemma 2 *For $f\in C^2[0,1]$:*

$$L_n=I-\frac{f(1)-f(0)}{2n}+O\Big(\frac{1}{n^2}\Big),\quad n\to\infty.$$

Proof. Denote $C=f(1)-f(0)$. Consider

$$\begin{aligned}I-L_n&=\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\Big(f(x)-f\Big(\frac{k}{n}\Big)\Big)\mathrm{d}x=\\&=\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\Big(f'\Big(\frac{k}{n}\Big)\Big(x-\frac{k}{n}\Big)+\frac{1}{2}f''(\theta_{kn}^x)\Big(x-\frac{k}{n}\Big)^2\Big)\mathrm{d}x=\\&=\frac{1}{2n^2}\sum_{k=0}^{n-1}f'\Big(\frac{k}{n}\Big)+r_n,\quad (1)\end{aligned}$$

where the remainder

$$|r_n|\le\frac{1}{2}\max|f''|\cdot\sum_{k=0}^{n-1}\frac{1}{3}\Big(x-\frac{k}{n}\Big)^3\Big|_{\frac{k}{n}}^{\frac{k+1}{n}}\le\frac{\mathrm{const}}{n^2}.$$

Here $\theta_{kn}^x \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$ are intermediate points from Taylor's theorem in the neighborhood of point $\frac{k}{n}$.

Similarly

$$\int_0^1 f' \mathrm{d}x - \frac{1}{n}\sum_{k=0}^{n-1} f'\Big(\frac{k}{n}\Big) = \sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f''(\widetilde{\theta}_{kn}^x)\Big(x-\frac{k}{n}\Big)\mathrm{d}x = O\Big(\frac{1}{n}\Big).$$

So in the right hand side of (1) we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f'\Big(\frac{k}{n}\Big) \to \int_0^1 f' \mathrm{d}x = C, \quad n\to\infty,$$

and the error in the right hand side of (1) is of order $O\big(\frac{1}{n^2}\big)$.

Thus from (1) we have

$$I - L_n = \frac{C}{2n} + O\Big(\frac{1}{n^2}\Big), \quad n\to\infty.$$

■

Now we return to solution of our problem. We have $U_n = L_n + \frac{C}{n}$. Put $x_n = L_n + \frac{kC}{3n}$, $k=1,2$. Then

$$x_n = I + \frac{C}{n}\Big(\frac{k}{3}-\frac{1}{2}\Big) + O\Big(\frac{1}{n^2}\Big).$$

For $k=1$ we have $\frac{k}{3}-\frac{1}{2}<0$, so $x_n < I$ for large enough n; for $k=2$ we have $\frac{k}{3}-\frac{1}{2}>0$, so $x_n > I$ for large enough n. Thus for large enough n

$$L_n + \frac{C}{3n} < I < L_n + \frac{2C}{3N}.$$

Grading scheme. For stating the problem as a double inequality 1p

For stating the idea of approaching the problem that might lead to a solution 1p

For establing the estimate $I = L_n + \dfrac{f(1)-f(0)}{2n} + O\left(\dfrac{1}{n}\right)$ 7p

Complete answer 1p