Problem MM 1682

1682. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Let n be a positive integer. Prove that

$$\sum_{j=1}^{n} \frac{1}{\binom{n}{j}} = \frac{n+1}{2^n} \sum_{j=0}^{n-1} \frac{2^j}{j+1}.$$

Solution: We first denote the left side of the above equation by L_n and the right side by R_n. Then we note that

$$R_{n+1} = \frac{n+2}{2^{n+1}} \left(\sum_{j=0}^{n-1} \frac{2^j}{j+1} + \frac{2^n}{n+1} \right) = \frac{n+2}{2^{n+1}} \left(\frac{2^n R_n}{n+1} + \frac{2^n}{n+1} \right) = \frac{n+2}{2(n+1)} (R_n + 1)$$

for all $n \geq 1$. Moreover, it is $L_1 = R_1 = 1$, and so, in order to complete the proof, it suffices to show that

$$L_{n+1} = \frac{n+2}{2(n+1)} (L_n + 1) \qquad (*)$$

for all $n \geq 1$. Indeed, it is easy to check that

$$\frac{1}{\binom{n+1}{j}} + \frac{1}{\binom{n+1}{j+1}} = \frac{n+2}{n+1} \cdot \frac{1}{\binom{n}{j}}$$

for all $n \geq j \geq 1$. By summing the above identity for $j = 1, 2, ..., n$, we get

$$\sum_{j=1}^{n} \frac{1}{\binom{n+1}{j}} + \sum_{j=1}^{n} \frac{1}{\binom{n+1}{j+1}} = \frac{n+2}{n+1} \cdot \sum_{j=1}^{n} \frac{1}{\binom{n}{j}};$$

that is,

$$(L_{n+1} - 1) + (L_{n+1} - \frac{1}{n+1}) = \frac{n+2}{n+1} L_n,$$

from which (*) follows immediately by solving for L_{n+1}. The proof is complete.

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