A solution to the problem H-761 of November's 2014 issue of The Fibonacci Quarterly

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The Problem. *Prove that*

$$\sum_{n\geq 1}\frac{1}{n}\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots\right)^2=\frac{\pi^2\ln 2}{6}-\frac{\ln^3 2}{3}-\frac{3}{4}\zeta(3).$$

Solution: We use the following definitions and special values for the dilogarithm and trilogarithm function respectively:

$$\mathrm{Li}_2(x)=-\int_0^x\frac{\ln(1-t)}{t}\,dt\left(=\sum_{n\geq 1}\frac{x^n}{n^2}\right),\tag{1}$$

$$\mathrm{Li}_3(x)=\int_0^x\frac{\mathrm{Li}_2(t)}{t}\,dt,\tag{2}$$

$$\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{\ln^2 2}{2},\tag{3}$$

$$\mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{21}{24}\zeta(3)+\frac{\ln^3 2}{6}-\frac{\pi^2\ln 2}{12}.\tag{4}$$

where ζ is the Riemann zeta function, (see [1], p.6, 1.16 and p.165, 6.12 for (3) and (4) respectively.)

We set $f_n:=\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\cdots$ so that the sum to be evaluated is

$$S:=\sum_{n\geq 1}\frac{f_n^2}{n}.$$

At first we observe that

$$f_n=\sum_{k\geq 1}\frac{(-1)^{k-1}}{n+k}=\sum_{k\geq 1}(-1)^{k+1}\int_0^1 x^{n+k-1}\,dx=\int_0^1 x^n\sum_{k\geq 0}(-x)^k\,dx=\int_0^1\frac{x^n}{1+x}\,dx,\tag{5}$$

so

$$0 < f_n < \frac{1}{n}. \tag{6}$$

Let us denote by H_n the $n-$th harmonic number.
We will use that

$$\sum_{n\geq 1}\frac{H_n}{n^2} = 2\zeta(3), \tag{7}$$

(see [2].) Summing by parts we have

$$\begin{aligned}
S &= \sum_{n\geq 1}(H_n - H_{n-1})\, f_n^2 = \lim_{n\to+\infty} H_n f_{n+1}^2 - H_0 f_1 - \sum_{n\geq 1} H_n(f_{n+1}+f_n)(f_{n+1}-f_n)\\
&\overset{(6)}{=} -\sum_{n\geq 1} H_n(f_{n+1}+f_n)(f_{n+1}-f_n)\\
&= \sum_{n\geq 1}\frac{H_{n+1}-\frac{1}{n+1}}{n+1}\left(\frac{1}{n+1}-\frac{2}{n+2}+\frac{2}{n+3}-\cdots\right)\\
&= \sum_{n\geq 2}\frac{H_n}{n}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right) - \sum_{n\geq 2}\frac{1}{n^2}\left(\frac{1}{n}-\frac{2}{n+1}+\frac{2}{n+2}-\cdots\right)\\
&\overset{(5)}{=} \sum_{n\geq 2}\frac{H_n}{n^2} - 2\sum_{n\geq 2}\frac{H_n}{n}\int_0^1\frac{x^n}{1+x}\,dx - \sum_{n\geq 2}\frac{1}{n^3} + 2\sum_{n\geq 2}\frac{1}{n^2}\int_0^1\frac{x^n}{1+x}\,dx\\
&\overset{(1),(7)}{=\!=} \zeta(3) - 2\int_0^1\frac{1}{1+x}\sum_{n\geq 1}\frac{H_n}{n}x^n\,dx + 2\int_0^1\frac{\mathrm{Li}_2(x)}{1+x}\,dx
\end{aligned}$$

It is easy to see that $\sum_{n\geq 1} H_n x^n = -\frac{\ln(1-x)}{1-x}$, so $\sum_{n\geq 1} H_n x^{n-1} = -\frac{\ln(1-x)}{x} - \frac{\ln(1-x)}{1-x}$ and finally $\sum_{n\geq 1}\frac{H_n}{n}x^n = \mathrm{Li}_2(x) + \frac{\ln^2(1-x)}{2}$ which means that

$$S = \zeta(3) - \int_0^1 \frac{\ln^2(1-x)}{1+x}\,dx.$$

In order to evaluate this last integral we integrate by parts as follows:

$$\begin{aligned}
\int\frac{\ln^2(1-x)}{1+x}\,dx &= \ln(1+x)\ln^2(1-x) + 2\int\frac{\ln(1+x)\ln(1-x)}{1-x}\,dx\\
&= \ln(1+x)\ln^2(1-x) + \int\frac{\ln\left(1-\frac{1-x}{2}\right)}{\frac{1-x}{2}}\ln(1-x)\,dx + 2\ln 2\int\frac{\ln(1-x)}{1-x}\,dx\\
&\overset{(1)}{=} \ln(1+x)\ln^2(1-x) + 2\mathrm{Li}_2\left(\frac{1-x}{2}\right)\ln(1-x) + \int\frac{\mathrm{Li}_2\left(\frac{1-x}{2}\right)}{\frac{1-x}{2}}\,dx - \ln 2\ln^2(1-x)\\
&\overset{(2)}{=} \ln\left(\frac{1+x}{2}\right)\ln^2(1-x) + 2\mathrm{Li}_2\left(\frac{1-x}{2}\right)\ln(1-x) - 2\mathrm{Li}_3\left(\frac{1-x}{2}\right) + c\\
&:= I(x) + c.
\end{aligned}$$

We therefore have

$$S = \zeta(3) + \lim_{x\to 0^+} I(x) - \lim_{x\to 1^-} I(x) \overset{(3),(4)}{=\!=} \frac{\pi^2 \ln 2}{6} - \frac{\ln^3 2}{3} - \frac{3}{4}\zeta(3).$$

□

References

[1] L. Lewin, *Polylogarithms and associated functions*, (1981), North Holland.

[2] *Mathproblems mathematical magazine*, Vol.4-2014, Issue 2, p.271, solution 2. Online: http://mathproblems-ks.com/?wpfb_dl=25