PROMYS Europe 2017

Application Problem Set

http://www.promys-europe.org

Please attempt each of the following problems. Though they can all be solved with no more than a standard high school mathematics background, most of the problems require considerably more ingenuity than is usually expected in high school. You should keep in mind that we do not expect you to find complete solutions to all of them. Rather, we are looking to see how you approach challenging problems. Here are a few suggestions:

- Think carefully about the meaning of each problem.
- Examine special cases, either through numerical examples or by drawing pictures.
- Be bold in making conjectures.
- Test your conjectures through further experimentation, and try to devise mathematical proofs to support the surviving ones.
- Can you solve special cases of a problem, or state and solve simpler but related problems?

If you think you know the answer to a question, but cannot prove that your answer is correct, tell us what kind of evidence you have found to support your belief. If you use books, articles, or websites in your explorations, be sure to cite your sources.

You may find that most of the problems require some patience. Do not rush through them. It is not unreasonable to spend a month or more thinking about the problems. It might be good strategy to devote most of your time to a small selection of problems which you find especially interesting. Be sure to tell us about progress you have made on problems not yet completely solved. **For each problem you solve, please justify your answer clearly and tell us how you arrived at your solution.**

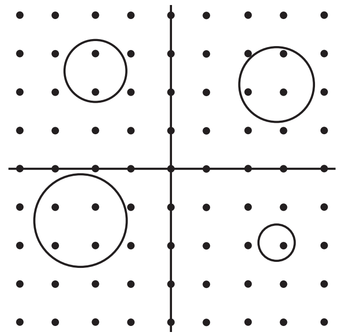
1. Calculate each of the following:

$$\begin{aligned} 1^3 + 5^3 + 3^3 &= ?? \\ 16^3 + 50^3 + 33^3 &= ?? \\ 166^3 + 500^3 + 333^3 &= ?? \\ 1666^3 + 5000^3 + 3333^3 &= ?? \end{aligned}$$

What do you see? Can you state and prove a generalization of your observations?

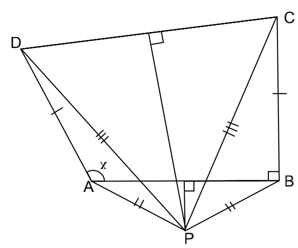
2. The *repeat* of a positive integer is obtained by writing it twice in a row (so, for example, the repeat of 2017 is 20172017). Is there a positive integer whose repeat is a perfect square? If so, how many such positive integers can you find?

3. A lattice point is a point (x, y) in the plane, both of whose coordinates are integers. It is easy to see that every lattice point can be surrounded by a small circle which excludes all other lattice points from its interior. It is not much harder to see that it is possible to draw a circle that has exactly two lattice points in its interior, or exactly 3, or exactly 4.



Do you think that for every positive integer n there is a circle in the plane containing exactly n lattice points in its interior? Justify your answer.

4. According to the Journal of Irreproducible Results, any obtuse angle is a right angle!



Here is their argument. Given the obtuse angle x, we make a quadrilateral $ABCD$ with $\angle DAB = x$, and $\angle ABC = 90°$, and $AD = BC$. Say the perpendicular bisector to DC meets the perpendicular bisector to AB at P. Then $PA = PB$ and $PC = PD$. So the triangles PAD and PBC have equal sides and are congruent. Thus $\angle PAD = \angle PBC$. But PAB is isosceles, hence $\angle PAB = \angle PBA$. Subtracting, gives $x = \angle PAD - \angle PAB = \angle PBC - \angle PBA = 90°$. This is a preposterous conclusion – just where is the mistake in the "proof" and why does the argument break down there?

5. A unit fraction is a fraction of the form $\frac{1}{n}$ where n is a positive integer. Note that the unit fraction $\frac{1}{11}$ can be written as the sum of two unit fractions in the following three ways:

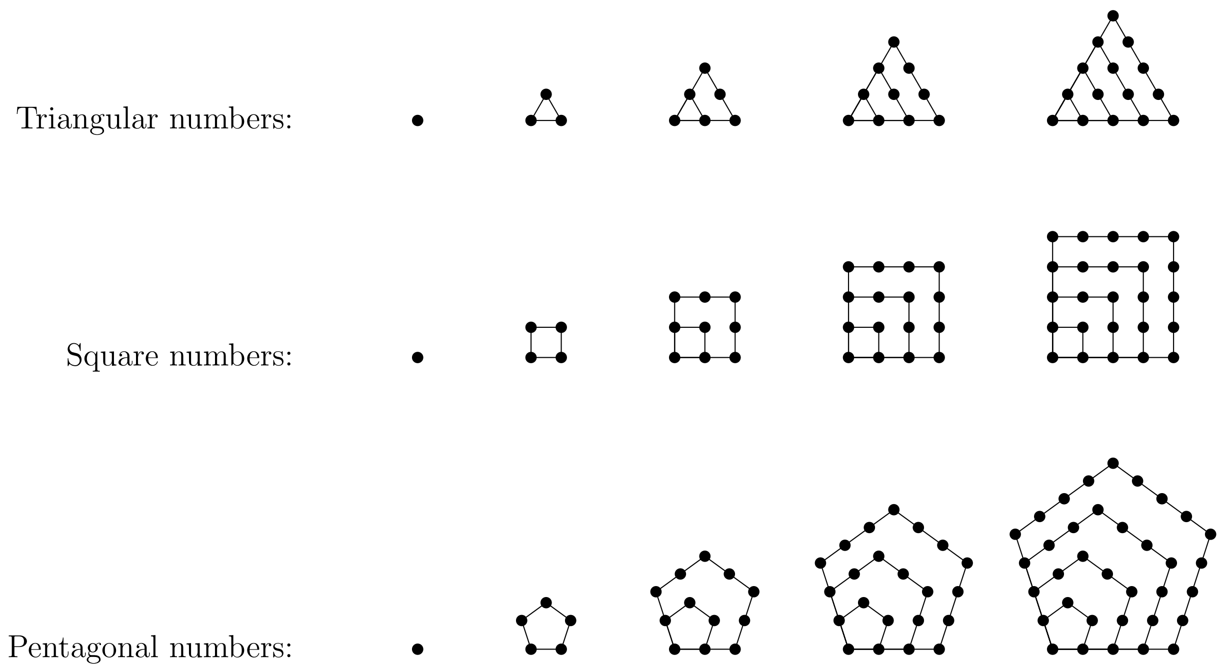
$$\frac{1}{11} = \frac{1}{12} + \frac{1}{132} = \frac{1}{22} + \frac{1}{22} = \frac{1}{132} + \frac{1}{12}.$$

Are there any other ways of decomposing $\frac{1}{11}$ into the sum of two unit fractions? In how many ways can we write $\frac{1}{60}$ as the sum of two unit fractions? More generally, in how many ways can the unit fraction $\frac{1}{n}$ be written as the sum of two unit fractions? In other words, how many ordered pairs (a, b) of positive integers a, b are there for which

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} \; ?$$

6. Let's agree to say that a positive integer is *prime-like* if it is not divisible by 2, 3, or 5. How many prime-like positive integers are there less than 100? less than 1000? A positive integer is *very prime-like* if it is not divisible by any prime less than 15. How many very prime-like positive integers are there less than 90000? Without giving an exact answer, can you say *approximately* how many very prime-like positive integers are less than 10^{10}? less than 10^{100}? Explain your reasoning as carefully as you can.

7. The triangular numbers are the numbers $1, 3, 6, 10, 15, \ldots$. The square numbers are the numbers $1, 4, 9, 16, 25, \ldots$. The pentagonal numbers are $1, 5, 12, 22, 35, \ldots$. The geometrical language is justified by the following diagrams:



a. What are the first five hexagonal numbers? What are the first five septagonal numbers? What are the first five r-gonal numbers? Give a formula for the nth triangular number. Give a formula for the nth square number. Give a formula for the nth pentagonal number. In general, give a formula for the nth r-gonal number.

b. How many numbers can you find that are simultaneously triangular and square? How many numbers can you find that are simultaneously square and pentagonal?

8. 10 people are to be divided into 3 committees, in such a way that every committee must have at least one member, and no person can serve on all three committees. (Note that we do not require everybody to serve on at least one committee.) In how many ways can this be done?

9. Let S be a set of positive real numbers. If S contains at least four distinct elements show that there are elements $x, y \in S$ such that

$$0 < \frac{x-y}{1+xy} < \sqrt{3}/3.$$

What can you say if S has at least 7 elements? What if S has at least n elements, where $n > 2$?

10. The tail of a giant kangaroo is tied to a pole in the ground by an infinitely stretchy elastic cord. A flea sits on the pole watching the kangaroo (hungrily). The kangaroo sees the flea, leaps into the air and lands one kilometre from the pole (with its tail still attached to the pole by the elastic cord). The flea gives chase and leaps into the air landing on the stretched elastic cord one centimetre from the pole. The kangaroo, seeing this, again leaps into the air and lands another kilometre away from the pole (i.e., a total of two kilometres from the pole). Undaunted, the flea bravely leaps into the air again, landing on the elastic cord one centimetre further along. Once again the kangaroo jumps another kilometre and the flea jumps another centimetre along the cord. If this continues indefinitely, will the flea ever catch up to the kangaroo? (Assume the earth is flat and extends infinitely far in all directions.)