Gliding principle

psi241

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This handouts will present one of theorem in euclidean geometry, gliding principle, and some problems that you may use this theorem to solve. This theorem can be used in problems about similar triangles or similar shapes. The reader should be familiar with directed angle and directed length before reading this article.

0 Notation

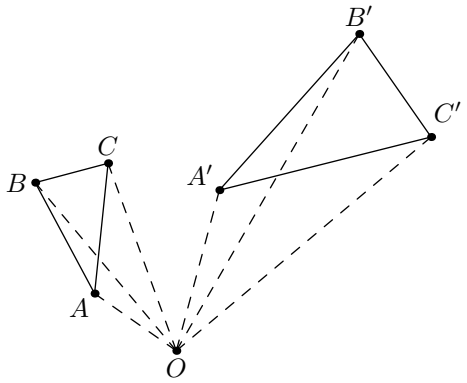
In this handout, For any polygons $P = P_1P_2...P_n, Q = Q_1Q_2...Q_n$, the symbol $P \overset{+}{\sim} Q$ means P and Q are similar and have same orientation. The symbol $P \overset{-}{\sim} Q$ means P and Q are similar but have different orientation. For both symbols, P_i, Q_i are corresponding vertex for all $i = 1, 2, ..., n$.

1 Spiral similarity

This section shows some theorem about spiral similarity that I need to prove the gliding principle.

Definition 1.1. (Spiral Similarity)

Spiral similarity is geometric transformation given by composition of a rotation with angle θ and a homothety with ratio k, both are about the center O. So, it sends a point A to A' where $\measuredangle AOA' = \theta$ and $OA' = k \cdot OA$.



From the definition, we can observe these corollaries.

Corollary 1.2. Spiral similarity preserves angles, ratios of lengths and orientations.

Corollary 1.3. Spiral similarity sends a polygon to another similar to itself.

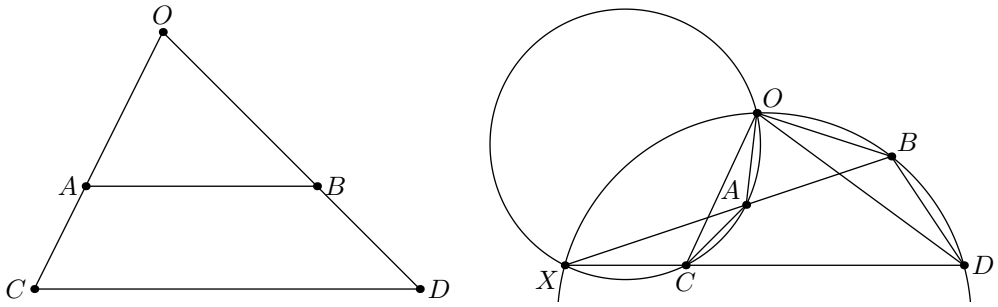
Corollary 1.3. follows corollary 1.2 because homothety and rotation preserve angles, ratios and send any shape to a similar one.

Next question : Given two similar polygons. Does there exist a spiral similarity that sends one to another?

Theorem 1.4. (Existence of spiral similarity between two segments)

If $ABDC$ is not a parallelogram, there exists a unique spiral similarity that sends $A \mapsto B, C \mapsto D$.

Proof.



If AB and CD are parallel, let O be the intersection of AC and BD. We see that $\measuredangle COD = \measuredangle AOB$ and $\frac{OD}{OC} = \frac{OB}{OA}$. So there exists a spiral similarity for this case.

If AB and CD intersect each other at X. Let O be another intersection of circumcircle of ACX and BDX. By angle chasing, $\triangle OAC \sim \triangle OBD$. Therefore, there exists a spiral similarity that sends A to B and C to D with angle $\measuredangle AOB$ and ratio $\frac{OB}{OA}$.

For any point $E \neq A, C$, if the spiral similarity that sends A to B and C to D, sends E to F, then $\measuredangle FBD = \measuredangle EAC$ and $\frac{FB}{BD} = \frac{EA}{AC}$. Hence, point F can be uniquely constructed. This concludes that there exists a unique spiral similarity. □

Next theorem follows from this one.

Theorem 1.5. (Existence of spiral similarity between similar polygon)

Let $P = P_1P_2...P_n$ and $Q = Q_1Q_2...Q_n$ be polygons such that $P \overset{+}{\sim} Q$ and they can not be transformed to each other by only translations. Then there exists a unique spiral similarity that sends P to Q.

Proof. By theorem 1.4, there exists unique spiral similarity with center O that sends P_1 to Q_1 and P_2 to Q_2 because $P_1Q_1Q_2P_2$ is not a parallelogram. For all integers $3 \leq i \leq n$, $P_1P_2P_i \overset{+}{\sim} Q_1Q_2Q_i$ since $P \overset{+}{\sim} Q$. So, from corollary 1.3, this spiral similarity sends P_i to Q_i for all $i = 1, 2, ..., n$. □

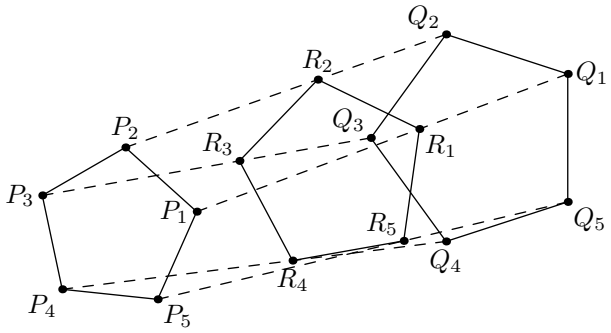
2 Gliding Principle

Next is our main theorem.

Theorem 2.1. (Gliding Principle)

Let $P = P_1P_2...P_n$ and $Q = Q_1Q_2...Q_n$ be polygons such that $P \overset{+}{\sim} Q$ and λ be a real number. Polygon $R_1R_2...R_n$ is given by $\frac{\overline{P_iR_i}}{\overline{P_iQ_i}} = \lambda$ for all $i = 1, 2, ..., n$. Then $R_1R_2...R_n \overset{+}{\sim} P_1P_2...P_n \overset{+}{\sim} Q_1Q_2...Q_n$.

This theorem is motivated by this observation: I draw lines joining all pairs of corresponding vertices. I "glide" each vertex of P with fixed velocity so that each vertex moves to Q in 1 second. For any $0 \leq t \leq 1$ second after points begin moving, what shape should the points form?

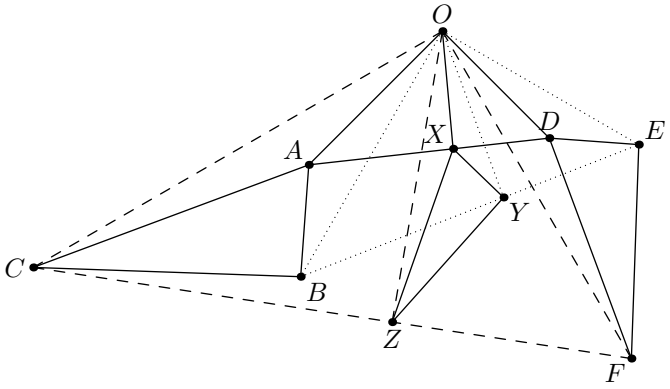


The answer is they should form the polygon R such that $P \overset{+}{\sim} Q \overset{+}{\sim} R$.

To prove this theorem, there is a complex-bashing proof which is easy to find. So, I will present synthetic proofs instead. The first proof uses spiral similarity.

Proof. We can prove this theorem by just proving the case $n = 3$, because if $P_1P_2...P_n \overset{+}{\sim} R_1R_2...R_n$ and $PP_iP_j \overset{+}{\sim} RR_iR_j$, then $PP_1P_2...P_n \overset{+}{\sim} RR_1R_2...R_n$. Let we call $P_1, P_2, P_3, Q_1, Q_2, Q_3, R_1, R_2, R_3$ by $A, B, C, D, E, F, X, Y, Z$ respectively.

In the case that $\triangle ABC$ can be translated to $\triangle DEF$ with vector $\vec{AD}$, $\triangle ABC$ can also be translated to $\triangle XYZ$ with vector $\vec{AX}$.



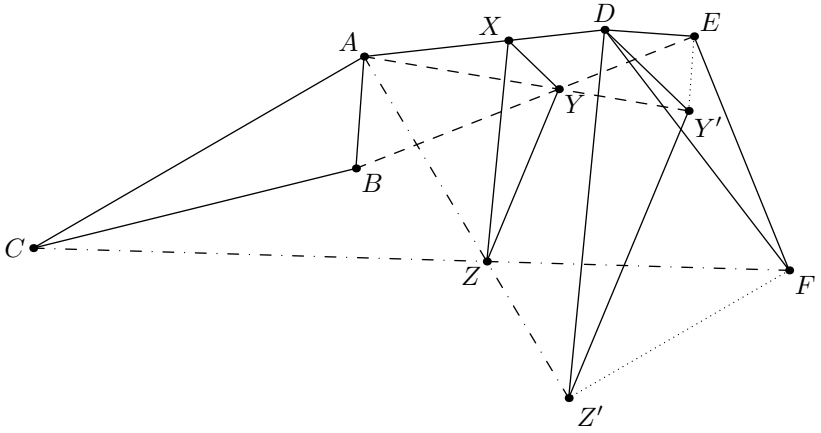
Otherwise, we observe that there exists a spiral similarity with center O (This is the reason why the orientation condition is necessary.) that maps $\triangle ABC \mapsto \triangle XYZ$. This gives that $\triangle OAD \overset{+}{\sim} \triangle OBE \overset{+}{\sim} \triangle OCF$. The ratios condition gives $\triangle OAX \overset{+}{\sim} \triangle OBY \overset{+}{\sim} \triangle OCZ$. Thus, There exists another spiral with center O sending $\triangle ABC \mapsto \triangle XYZ$.

Therefore, $\triangle ABC, \triangle DEF, \triangle XYZ$ are similar to each other and all have the same orientation. $\square$

Here is an alternative proof.

Proof. As in the first proof, It suffices to prove this theorem for $n = 3$.

Define $A, B, C, D, E, F, X, Y, Z$ as same as the first proof.



Let AY intersect a line passing through E and parallel to AB at Y'; and AZ intersect a line passing through F parallel to AC at Z'. Since $\frac{Y'E}{AB} = \frac{Z'F}{AC}$ and $\frac{DE}{AB} = \frac{DF}{AC}$, so $\frac{Y'E}{Z'F} = \frac{DE}{DF}$.
Because $\triangle ABC \overset{+}{\sim} \triangle DEF$, we obtain that $\measuredangle Y'ED = \measuredangle(AB, DE) = \measuredangle(AC, DF) = \measuredangle Z'FD$
Hence, we get that $\triangle DEY' \overset{+}{\sim} DFZ'$, so as $\triangle DEF \overset{+}{\sim} \triangle DY'Z'$.
Now, we have a homothety with center A which sends $\triangle DY'Z'$ to $\triangle XYZ$. So, we are done. □

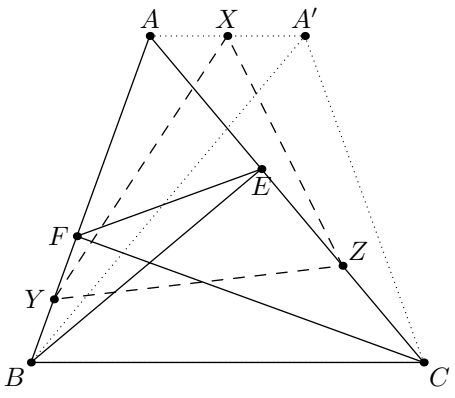
3 Examples

I will present applications of gliding principle with these two problems.

Example 3.1.

Let $\triangle ABC$ be an acute scalene triangle. BE, CF are altitudes of this triangle. Y, Z are midpoints of BF, CE respectively. X is a point such that $\triangle XYZ \overset{-}{\sim} \triangle ACB$. Prove that $BX = CX$.

First, let's point out some ideas. We see two similar triangles with same orientation, $\triangle AEF \overset{+}{\sim} \triangle XZY$. Furthermore, Y, Z are defined by being the midpoints of BF, CE that we can eaily relate them to $\triangle ABC$. Hence, we can glide it to other triangle. This is an idea to approach this problem.

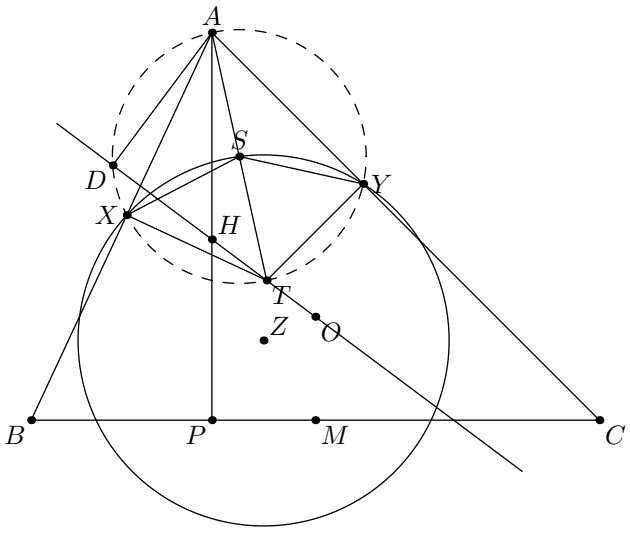


Proof. Let A' be the reflection of A across X. We see that $\triangle XZY \overset{+}{\sim} \triangle AEF \overset{-}{\sim} \triangle ABC$ and $\frac{\overline{BF}}{\overline{YF}} = \frac{\overline{CE}}{\overline{ZE}} = \frac{\overline{A'A}}{\overline{XA}} = 2$. By gliding principle, $\triangle A'BC \overset{+}{\sim} \triangle AFE \overset{-}{\sim} ABC$. Therefore A' must be the reflcetion of A across the perpendicular bisector of BC. This implies that X lies on perpendicular bisector of BC or $BX = CX$. □

Example 3.2. (IMO shortlist 2016/G5)

Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcenter and the orthocenter) of an acute scalene triangle ABC. A circle ω with center S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcenter of triangle XSY is equidistant from P and M.

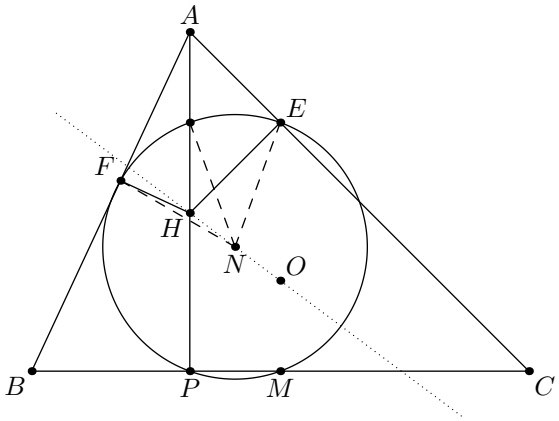
Let Z be the circumcenter of XSY. First thing, Points X, Y, S are only points which related to Z. By angle chasing, each angle XYZ is always same so gliding principle may help to show that another vertex Z of XYZ lies on a line not depend on choice of S by proving only two cases. Next we need to find propoties of X, Y and how they change depended on S. To deal with S easier, homothety S with ratio 2 and center A gives that new point lies on the Euler line. This leads to the below solution. This leads to the below solution.



Proof. Let AS intersect the Euler line at T and H, O be the orthocenter and circumcenter of $\triangle ABC$. We see that AT is diameter of $\odot AXY$. Note that $TX \perp AB, TY \perp AC$. Let Z be the circumcenter of $\triangle XSY$. Because S can be chosen from only on the perpendicular bisector of AD, we see that we can define point X, Y, S, Z from point T which can be chosen arbitrarily from the Euler line. We will show that Z always lies on perpendicular bisector of PM

Case I $T = H$

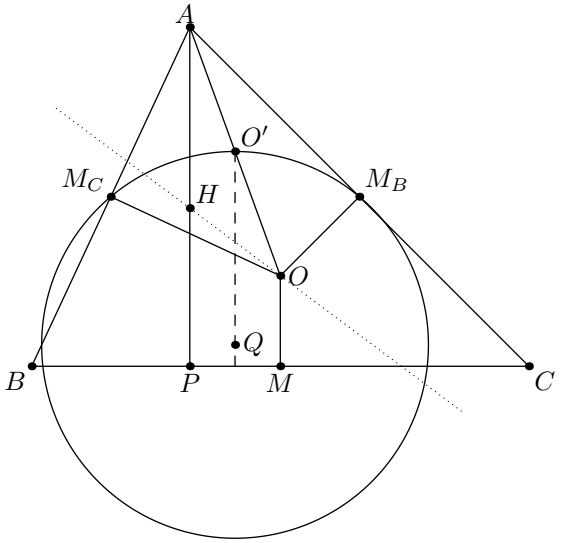
We see that $X = F$,$Y = E$ where BE, CF are altitudes of $\triangle ABC$ and S is midpoint of AH. Clearly, circumcenter of $\triangle XSY$ is nine-point center of $\triangle ABC$, call this point N. We've known that N lies on perpendicular bisector of PM.



Case II $T = O$

We see that $X = M_C, Y = M_B$ where M_C, M_B are midpoints of AB, AC.

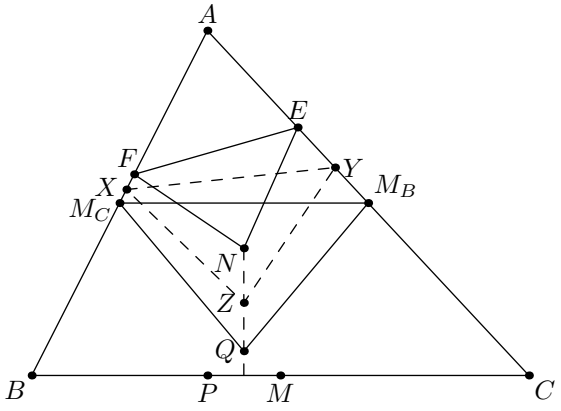
Let O', Q be the circumcenters of $\triangle AM_BM_C, \triangle O'M_BM_C$. In this case, $O'Q$ is perpendicular to M_CM_B. so $O'Q$ is a line equidistant to altitude from A and the perpendicular line from O to BC. So, Q lies on the perpendicular bisector of PM.



Case III $T \neq H, O$

Because $\triangle XZY$ is always an isosceles triangle with $\measuredangle XZY = 2 \cdot \measuredangle XSY = 4 \cdot \measuredangle BAC$. So, $\triangle XZY \overset{+}{\sim} \triangle FNE \overset{+}{\sim} \triangle M_CQM_B$. We see that $\frac{\overline{XF}}{\overline{M_CF}} = \frac{\overline{YE}}{\overline{M_BE}} = \frac{\overline{TH}}{\overline{OH}}$. Let Z' be the point on the perpendicular bisector of PM such that $\frac{\overline{Z'N}}{\overline{QN}} = \frac{\overline{TH}}{\overline{OH}}$. By gliding principle, $\triangle XZ'Y \overset{+}{\sim} \triangle FNE \overset{+}{\sim} \triangle M_CQM_B$.

This implies that $Z = Z'$. In summary, Z lies on the perpendicular bisector of PM.



□

4 Problems

Problem 1. (ERIQ lemma) Let $ABCD$ be a quadrilateral. E, F be a point on AB, CD respectively such that $\frac{\overline{AE}}{\overline{EB}} = \frac{\overline{DF}}{\overline{FC}}$. Let X, Y, Z respectively be a point on AB, CD, EF such that

$$\frac{\overline{AX}}{\overline{XD}} = \frac{\overline{BY}}{\overline{YC}} = \frac{\overline{EZ}}{\overline{ZF}}.$$

Show that X, Y, Z collinear.

Problem 2. (IMO shortlist 2018/G2) Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the line PB and PC, respectively, so that B lies on segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that quadrilateral $APXY$ is cyclic.

Problem 3. Let $ABCD$ be a cyclic quadrilateral. F, G, M, N are midpoints of AD, BC, AC, BD respectively. K, L are the feet of altitudes from E to AD, BC respectively. P, Q are points such that quadrilaterals $FELP$ and $GEKQ$ are parallelograms. Prove that $MNPQ$ is an isosceles trapizoid.

Problem 4. (IMO shortlist 2006/G3) Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Problem 5. (Russia 2001 Grade 9/3) Let N be a point on the longest side AC of a triangle ABC. The perpendicular bisectors of AN and NC intersect AB and BC respectively at K and M. Prove that the circumcenter O of $\triangle ABC$ lies on the circumcircle of triangle KBM.

Problem 6. (IMO shortlist 2009/G6) Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

Problem 7. (IMO shortlist 2017/G3) Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

Problem 8. (IMO 2019/6) Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q. Prove that lines DI and PQ meet on the line through A perpendicular to AI.

Hints are given in the next page.

5 Hints

These are hints of the problems in previous section.

1. Degenerate triangle

2. Use condition about angle to construct cyclic quadrilateral. Then, glide them to $APXY$

3. Because M, N are midpoints. Try to make P, Q be midpoint of some segment. Then use gliding to conclude the result.

4. There are given three similar triangle with same orientation. Consider midpoint of some side of all given triangle.

5. When we move N to A, C, we can find an angle in $\triangle KOM$ easily. These two triangles, $\triangle KOM$ when $N = A$ and $N = C$, are similar to each others. For other point N, we can use gliding principle with it.

6. You need to find two similar triangles and relation between $P, H_1, H_2, O_1, O_2, E_1, E_2$, then delete A, B, C, D. Find a pair of similar triangle to use gliding principle.

7. Let AD, BE, CF be altitude from A, B, C respectively. It's easy to see that $\triangle PQH \overset{+}{\sim} \triangle AEF \overset{-}{\sim} \triangle ABC$. We try to glide the circumcenter of PQH. Try this observation: Glide circumcenter of these triangle. $\triangle AEF \overset{+}{\sim} \triangle HP'Q' \overset{+}{\sim} \triangle A'BC$ where D is midpoint of A, A' and P', Q' define to relate with P, Q and to satisfy some condition.

8.It is hard to find properties of line PQ unless you construct the correct point on that line. Then you will find similar triangles to use gliding principle.

References

[1] Yufei Zhao. *Three Lemmas in Geometry*, 2010.

[2] Nguyen Hoang Son. *E.R.I.Q lemma and applications.*

[3] `https://artofproblemsolving.com/community/q1h1671271p10723286`

[4] `https://artofproblemsolving.com/community/c6h355793p11311141`