We show that for each i with $1 \le i \le n$ we have $F(i) = i + 1$, so that in particular $F(n) = n + 1$. We use induction on i.

For $i = 1$ we have

$$F(i) \;=\; F(1) \;=\; \left\lceil \frac{n^2+2n+2}{n^2+n+1} \right\rceil \;=\; \left\lceil 1 + \frac{n+1}{n^2+n+1} \right\rceil .$$

Thus

$$1 \;<\; F(1) \;=\; \left\lceil 1 + \frac{n+1}{n^2+n+1} \right\rceil \le \left\lceil 1 + \frac{n+1}{n^2+n} \right\rceil \;=\; \left\lceil 1 + \frac{1}{n} \right\rceil \;=\; 2\,,$$

from which we see that $F(1) = 2$, as required.

If we assume validity for $i = m - 1$ (that is, $F(m-1) = m$, where $2 \le m \le n$) then

$$\begin{aligned} F(m) \;&=\; \left\lceil \frac{n^2+2n+m+1}{n^2+n+m} \cdot m \right\rceil \;=\; \left\lceil m + \frac{(n+1)m}{n^2+n+m} \right\rceil \\ &\le\; \left\lceil m + \frac{(n+1)m}{n^2+n} \right\rceil \;=\; \left\lceil m + \frac{m}{n} \right\rceil \le \lceil m+1 \rceil \;=\; m+1\,, \end{aligned}$$

but as $\dfrac{(n+1)m}{n^2+n+m} \ge 0$, we have $m < \left\lceil m + \dfrac{(n+1)m}{n^2+n+m} \right\rceil$.

Thus we have $m < F(m) \le m + 1$, showing that $F(m) = m + 1$, completing the induction step and proving the claim.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; DAVID DOSTER, Choate Rosemary Hall, Wallingford, Connecticut, USA; FLORIAN HERZIG, student, Cambridge, UK; RICHARD I. HESS, Rancho Palos Verdes, California, USA; JOEL SCHLOSBERG, student, Bayside, NY, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer.

2338. [1998: 234] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose $ABCD$ is a convex cyclic quadrilateral, and P is the intersection of the diagonals AC and BD. Let I_1, I_2, I_3 and I_4 be the incentres of triangles PAB, PBC, PCD and PDA respectively. Suppose that I_1, I_2, I_3 and I_4 are concyclic.

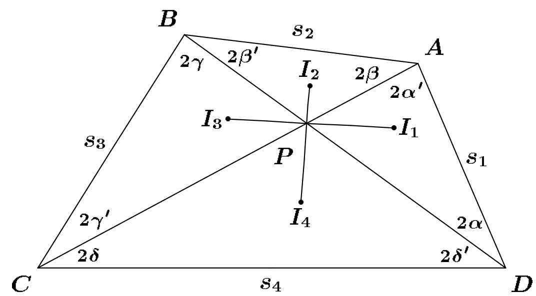
Prove that $ABCD$ has an incircle.

Solution by Peter Y. Woo, Biola University, La Miranda, California, USA.

$ABCD$ does not have to be cyclic. More precisely,

When a convex quadrilateral is subdivided into 4 triangles by its two diagonals, then the incentres of the 4 triangles are concyclic if and only if the quadrilateral has an incircle.

Notation. Let P be the point where the diagonals intersect and let the triangles be T_1, T_2, T_3, T_4 (labelled counterclockwise as in the figure), with the respective incentres I_1, I_2, I_3, I_4. Denote the 8 angles formed by the diagonals with the four sides by 2α, $2\alpha'$, 2β, $2\beta'$, 2γ, $2\gamma'$, 2δ, $2\delta'$ (counterclockwise with 2α, $2\alpha'$ in T_1, etc.).



Step 1. In the usual notation (used only here in step 1) for $\triangle ABC$ with sides a, b, c, incentre I, and semiperimeter $s = \frac{a+b+c}{2}$, AI satisfies

$$AI^2 \;=\; bc \tan\frac{B}{2}\tan\frac{C}{2}\,.$$

The proof follows from familiar formulas. In E.W. Hobson's *Treatise on Plane and Advanced Trigonometry*, for example, in section 123 the author shows that

$$AI \;=\; \frac{r}{\sin\frac{A}{2}}\,,\quad \tan\frac{B}{2} \;=\; \frac{r}{s-b}\,,\quad \tan\frac{C}{2} \;=\; \frac{r}{s-c}\,,$$

$$\text{and}\quad \sin^2\frac{A}{2} \;=\; \frac{(s-b)(s-c)}{bc}\,,$$

which, when combined, proves the claim.

Step 2. I_1, I_2, I_3, I_4 are concyclic if and only if

$$\tan\alpha\tan\alpha'\tan\gamma\tan\gamma' \;=\; \tan\beta\tan\beta'\tan\delta\tan\delta'\,.$$

Proof. Since PI_1 and PI_3 bisect vertically opposite angles, as do PI_2 and PI_4, the Intersecting Chords Theorem says that I_1, I_2, I_3, I_4 are concyclic if and only if $PI_1 \cdot PI_3 = PI_2 \cdot PI_4$. The desired equality then follows from step 1.

Step 3. A quadrilateral has an incircle if and only if the sum of one pair of opposite sides equals the sum of the other. This is a standard result of elementary geometry. See, for example, Nathan Altshiller Court's *College Geometry*, p. 135.

Step 4. Let $ABCD$ be any quadrilateral (that is, any four points, no three collinear), *I, I' be incentres of $\triangle ABC$ and $\triangle ADC$, and IN, $I'N'$ be* *perpendiculars to the diagonal AC from I and I'. Then $CN \geq CN'$ if and only if $AD + BC \geq AB + CD$, with equality for both or for neither.*

Proof. $CB - AB = CN - AN$ [because $CN = s - c$ and $AN = s - a$ in the notation of step 1] and $AD - CD = AN' - CN'$. Add these two equalities [noting that $AN = AC - CN$ and $AN' = AC - CN'$].

Step 5. $AD + BC \geq AB + CD$ if and only if

$$\tan\angle BAI \tan\angle DCI' \;\geq\; \tan\angle BCI \tan\angle DAI' ,$$

with equality for both or for neither.

Proof. This follows from step 4 since we have $\tan\angle BAI = \frac{IN}{AC-CN}$, $\tan\angle DCI' = \frac{I'N'}{CN'}$, $\tan\angle BCI = \frac{IN}{CN}$, and $\tan\angle DAI' = \frac{I'N'}{AC-CN'}$. [Note that the incentres here generally do not coincide with those of the main result. The key observation is that IA (for example) bisects the angle between a diagonal and side of the quadrilateral, while the angles α, α', etc. in step 2 each are equal to half the angle between a diagonal and side.]

Proof of the main result. If $ABCD$ has an incircle then $AD + BC = AB + CD$ (step 3), so that $\tan\alpha\tan\gamma = \tan\beta'\tan\delta'$ and $\tan\alpha'\tan\gamma' = \tan\beta\tan\delta$ (step 5). By step 2, I_1, I_2, I_3, I_4 are therefore concyclic. On the other hand, if $ABCD$ does not circumscribe some circle, then let s_i be the side of T_i opposite P (for $i = 1, 2, 3, 4$). Without loss of generality, assume $s_2 + s_4 > s_1 + s_3$. Then by step 5, $\tan\alpha\tan\gamma > \tan\beta'\tan\delta'$ and $\tan\alpha'\tan\gamma' > \tan\beta\tan\delta$, so that by step 2, I_1, I_2, I_3, I_4 are not concyclic.

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All solvers except Woo and Janous proved the theorem as stated (with $ABCD$ cyclic). Janous remembers having seen the stronger version before, but he did not recall the reference. Can any reader provide a reference?

2340. [1998: 235] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let $\lambda > 0$ be a real number and a, b, c be the sides of a triangle. Prove that

$$\prod_{\text{cyclic}} \frac{s + \lambda a}{s - a} \geq (2\lambda + 3)^3.$$

[As usual s denotes the semiperimeter.]