proper axiom or is inferred by modus ponens on S_j and S_k for some j and k less than i, or is in Δ.

If Δ is the empty set, we write $\vdash_T Q$ and say that Q is a *theorem of* T.

The only difference between a deduction in the predicate calculus and a deduction in T is that certain statements that would be labeled as assumptions in the predicate calculus are labeled as proper axioms in T.

THEOREM 1. *Let* T *be a first order theory.* $\vdash_T Q$ *if and only if there is a finite set* $\{P_1, \ldots, P_k\}$ *of proper axioms of* T *such that* $P_1, \ldots, P_k \vdash Q$ *and each step of the deduction is a formula of* T.

Proof. Suppose $\vdash_T Q$. Let $S_1, \ldots, S_n$ be a proof of Q in T. Let $P_1, \ldots, P_k$ be the steps that are labeled as proper axioms. Relabel each P_i as an assumption. Then $S_1, \ldots, S_n$ is a deduction of Q from $\{P_1, \ldots, P_k\}$ in the predicate calculus, i.e., $P_1, \ldots, P_k \vdash Q$. Now suppose there is a finite set $\{P_1, \ldots, P_k\}$ of proper axioms of T such that $P_1, \ldots, P_k \vdash Q$. Let $S_1, \ldots, S_n$ be a deduction of Q from $\{P_1, \ldots, P_k\}$ in the predicate calculus. Relabel each P_i as a proper axiom. Then $S_1, \ldots, S_n$ is a proof of Q in T, i.e., $\vdash_T Q$.

We give some examples of first order theories. With one exception, all the examples are from modern algebra. We do not expect the reader to be an expert in modern algebra before or after he reads these examples. He should read all the examples, and return to a particular one as the need arises.

EXAMPLE 2. (Compare Example 7.1.) The set of proper symbols of the theory L of *linearly ordered sets* is $\{=, <\}$. The proper axioms of L are

L1. $\forall x \sim(x < x)$

L2. $\forall x \forall y \forall z(x < y \wedge y < z \rightarrow x < z)$

L3. $\forall x \forall y(x < y \vee x = y \vee y < x)$

Recall that $=$ is the first 2-place predicate symbol, i.e., $r = s$ is an abbreviation for $\alpha\#\#(r, s)$. $<$ is the second 2-place predicate symbol, i.e., $r < s$ is an abbreviation for $\alpha\#\#|(r, s)$. P is a formula of L if and only if every proper symbol that occurs in P is in $\{=, <\}$. The equality axioms of L are

E1. $\forall x(x = x)$

E2=. $\forall x_1 \forall x_2 \forall y_1 \forall y_2(x_1 = y_1 \wedge x_2 = y_2 .\rightarrow. x_1 = x_2 \leftrightarrow y_1 = y_2)$

E2<. $\forall x_1 \forall x_2 \forall y_1 \forall y_2(x_1 = y_1 \wedge x_2 = y_2 .\rightarrow. x_1 < x_2 \leftrightarrow y_1 < y_2)$